Stat 134 MGF homework problems

Problem 1 (properties of the moment generating function). We defined the moment generating function (m.g.f.) of a random variable X as $\psi_X(t) = E(e^{Xt})$ which is a function of t (note: I sometimes use $\psi_X(t)$ instead of $M_X(t)$ as the name of a moment generating function). More explicitly, it has one of the two forms:

$$\psi_X(t) = E(e^{Xt}) = \sum_{\text{all } x} e^{xt} P(X = x) \qquad \text{or} \qquad \psi_X(t) = E(e^{Xt}) = \int_{-\infty}^{\infty} e^{xt} f_X(x)\, dx,$$

if X is a *discrete* or *continuous* random variable, respectively.

(a) Show that, for any r.v. X, it is the case that $\psi_X(0) = 1$. (So the m.g.f. is always defined at $t = 0$.)

(b) Show that if $Y = aX + b$, where a and b are constants, then $\psi_Y(t) = e^{bt}\psi_X(at)$.

(c) Show that if the random variables X and Y are *independent* and $Z = X + Y$, then $\psi_Z(t) = \psi_X(t)\psi_Y(t)$. *Hint:* you have to use the fact that if two random variables X and Y are independent and f and g are two functions, then the two new random variables $U = f(X)$ and $V = g(Y)$ are also independent.

Problem 2 Another property of the moment generating function is that if two random variables have the same m.g.f., then they have the same p.d.f. (we say that the mapping p.d.f. $\mapsto$ m.g.f. is one-to-one). In class we showed that if we have $X \sim \text{Gamma}(\alpha, \lambda)$, then $\psi_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$. Use these facts and part **(c)** of the previous problem to show that if $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta, \lambda)$ are independent random variables and we define $Z = X + Y$, then $Z \sim \text{Gamma}(\alpha + \beta, \lambda)$.

Problem 3 (m.g.f. of Binomial random variables). We proved that the m.g.f. $\psi_X(t)$ "generates" the moments of the random variable X by *differentiation*, and computation at $t = 0$ (rather than by integration or summation, which is typically harder). For example,

$$\psi'_X(0) = E(X), \quad \psi''_X(0) = E(X^2), \quad \ldots, \quad \psi_X^{(n)}(0) = E(X^n),$$

where the superscript $^{(n)}$ indicates the n^{th} derivative.

(a) Assume that $X \sim \text{Binomial}(n, p)$, i.e. with p.f. $f_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, for $k = 0, 1, \ldots, n$.
Find the function $\psi_X(t)$, in terms of the parameters n and p.

Hint: You will need to use the binomial theorem, which states that $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a + b)^n$.

(b) Find $E(X)$ and $\text{Var}(X)$ using the method described above.

Note that this is *much* simpler than computing, for example, $E(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$.

Problem 4 (m.g.f. of Normal random variables).

(a) Consider a *standard* Normal random variable $Z \sim \mathcal{N}(0,1)$, and compute its m.g.f. ψ_Z. *Hint:* You need to "complete the square" at the exponent: $\frac{z^2}{2} - tz = \frac{1}{2}(z^2 - 2tz + t^2) - \frac{t^2}{2} = \frac{1}{2}(z-t)^2 - \frac{t^2}{2}$, and then perform a change of variable in the integral.

(b) Now use part (b) of Problem 1 to compute the m.g.f. of a *generic* Normal r.v. $X \sim N(\mu, \sigma^2)$.

(c) Finally, use part (c) of Problem 1 to show that if $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent and we define $Z = X + Y$, then $Z \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Remark: it should be clear by now that using the moment generating functions to prove the above fact is a much simpler method than computing the convolution between two Normal p.d.f.'s!

Problem 5 Suppose that X is a random variable for which the m.g.f. is as follows:

$$\psi_X(t) = \frac{1}{6}\Big(4 + e^t + e^{-2t}\Big).$$

Find the probability distribution of X. *Hint:* It is a simple discrete distribution for example, pmf $P(X=0) = .3$, $P(X=1) = .7$.