Final Review Sheet Answers

STAT 134: Adam Lucas

November 16, 2020

The proofs/calculations of most exercises here are omitted. Again, refer to your notes if unsure; all of the theoretical results have been discussed in lecture notes or in the textbook.

1 Some common aspects of densities

1. *Constant and variable parts of densities:*

   $X \sim$ Gamma $(4, 6)$; $\int_0^\infty 5x^3 e^{-6x} dx = 5(\frac{\Gamma(4)}{6^4})$

2. *Independence and dependence:*

   In the continuous case, we may show that $f_X(x, y) = f_X(x)f_Y(y)$, **and** that the support of X and Y do not depend on each other (i.e., for all (x, y), we have that $f_{X,Y}(x, y) > 0$ if and only if $f_X(X) > 0$ and $f_Y(y) > 0$.)

   An easy way to show that two variables are dependent is to find x, y such that $P(Y = y) > 0$ but $P(Y = y|X = x) = 0$. The idea is to find a value of X that makes a particular value of Y impossible.

3. *Working with densities:*

   (a) The CDF of X is given by $F_X(x) = \int_{-\infty}^x f_X(t)dt$. By the fundamental theorem of calculus, it follows that $f_X(x) = F'_X(x)$.

   (b) $f_X(x) = \int_{-\infty}^\infty f_{X,Y}(x, y)dy$

2 Relationships Between Distributions

1. cX, where $X \sim$ Exp (λ), $c > 0$:

   Exp $(\frac{\lambda}{c})$. The methods we have used include using the CDF of X, using the change of variable method, and using the MGF of X and properties of MGFs.

2. Consider $X = U_{(3)}$ and $Y = U_{(7)}$, the 3rd and 7th order statistics of 10 iid Unif $(0, 1)$ RVs. What is the distribution of X/Y?

   We proved in lecture that for this type of example (ratio of joint Uniform (0,1) order statistics, where the numerator is the smaller one), the resulting distribution is a Beta. In particular, this example follows the Beta (3,4) distribution.

3. $X^2 + Y^2$, where X, Y are independent standard Normal. What is $\sqrt{X^2 + Y^2}$?

   Exp $(\frac{1}{2})$; standard Rayleigh

3 Symmetry

1. Under some conditions, we can quickly recognize the expectation of a random variable X to be zero. What are they?

   The distribution/density of X must be symmetric about the origin, and $\mathbb{E}(|X|) < \infty$; i.e. the expectation must be defined.

2. Let X, Y be independent $\mathcal{N}(0, \sigma^2)$ random variables. Without using Φ, find $P(X + 2Y > 0, X > 0)$. What is the reason behind this answer?

   Using the rotational symmetry of the joint distribution of (X, Y),

$$P(X + 2Y > 0, X > 0) = \frac{\frac{\pi}{2} + \arctan(\frac{1}{2})}{2\pi}$$

4 Simplifying an infinite sum

1. Let Y have density $f_Y(y) = \frac{\lambda}{2} e^{-\lambda|y|}$, for $y \in \mathbb{R}$. With little computation, find $\mathbb{E}(|Y|)$ and $\mathbb{E}(Y)$.

   If we look at the graph of the density of Y, or through the change-of-variable formula, we observe that $|Y| \sim$ Exp (λ). Thus $\mathbb{E}(|Y|) = \frac{1}{\lambda}$, and $\mathbb{E}(Y) = 0$ by symmetry.

5 Could You Rephrase That?

1. Let $X \sim$ Gamma (r, λ), where r is an integer. Use what we know about the Poisson process to obtain the CDF of X.

$$\begin{aligned} P(X < t) &= P(N_t \geq r) \\ &= 1 - P(N_t < r) \\ &= 1 - \sum_{k=0}^{r-1} e^{\lambda t} \frac{(\lambda t)^k}{k!} \end{aligned}$$

2. Let X be an arbitrary random variable with an invertible CDF F_X. What is the random variable formed by $F_X(X)$?

   Let $Z = F_X(X)$. Then,

$$\begin{aligned} F_Z(z) &= P(Z \leq z) \\ &= P(F_X(X) \leq z) \\ &= P(F_X^{-1}(F_X(X)) \leq F_X^{-1}(z)) \\ &= P(X \leq F_X^{-1}(z)) \\ &= F_X(F_X^{-1}(z)) \\ &= z, \quad z \in [0, 1] \end{aligned}$$

   We conclude that $Z \sim$ Unif (0,1).

6 Some Useful Results

1. Let $X \sim \text{Exp}(\lambda_X)$, $Y \sim \text{Exp}(\lambda_Y)$. What is $P(X < Y)$? $\frac{\lambda_X}{\lambda_X + \lambda_Y}$

2. Find the distribution of $\min\{X_1, X_2, \ldots, X_n\}$, where the X_i's are independent $\text{Exp}(\lambda)$ variables.

   Let M denote the minimum. Then $M \sim \text{Exp}(n\lambda)$. Note this result generalizes to the case where the rates are all different; you simply add the rate parameters.

3. Suppose cars and trucks arrive at a bridge according to independent Poisson processes with rates λ_c and λ_r per minute respectively. Given that n vehicles arrive in t minutes, what is the distribution of $N_{c,t}$, the number of cars to arrive by time t?

   The easiest way to proceed here is using the conditional probability rule. We find that $N_{c,t} | N_t = n \sim \text{Binom}\left(n, \frac{\lambda_c}{\lambda_c + \lambda_r}\right)$.