Stat 134 Spring 2019: Practice Midterm Questions: Solutions

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1. Flip a coin 8 times. Given that there are 3 heads, find the probability that:

   (a) The heads are not consecutive.

   Since we're looking for a probability, let's figure out the denominator and numerator separately. The total number of ways to place 3 heads in 8 coin tosses is going to be $\binom{8}{3}$, where order does not matter. So, $\binom{8}{3}$ is our denominator.

   To make sure no head is next to another head, let's lay down the tails first. With 5 tails, 6 slots are created where one head can go in.

   Note that different from problem 2 on Feb. 21st worksheet, once a head goes into a slot, neither does it create an extra slot nor can other heads go into the same slot.

   So our answer is

$$\frac{\binom{6}{3}}{\binom{8}{3}}$$

   (b) The heads are all next to each other.

   Now if we all of the heads to be next to each other, we have to place all three of them into one slot. To do this, let's consider all three of them as a group. Now I just need to pick one slot to place this **one** group. So our numerator is $\binom{6}{1}$. Hence, our final answer is

$$\frac{\binom{6}{1}}{\binom{8}{3}}$$

   (c) There are 2 heads next to each other, and 1 other head by itself.

   Let's consider 2 heads as a group, and the one head left as another group. Now we need to pick 2 slots to place the two groups, and that gives us $\binom{6}{2}$. But keep in mind that we have to pick one slot from the 2 for the group of 2 heads to go in! The idea behind this is that the two groups are **not identical**, just like triples and pairs are not identical in a poker hand. So our numerator is $\binom{6}{2}\binom{2}{1}$. Our final answer is

$$\frac{\binom{6}{2}\binom{2}{1}}{\binom{8}{3}}$$

2. Suppose a coin lands heads with probability p. Toss the coin until you observe three heads. Given your third head occurred on toss n, what is the chance your first head landed on toss k, for $k \in \{1, 2, \ldots, n-2\}$?

   Let X_1, X_3 denote the trial on which you get your first and third head, respectively. Note $X_1 \sim \text{Geom}\,(p)$ on $\{1, 2, \ldots\}$ and $X_3 \sim \text{NegBin}\,(3, p)$ on $\{3, 4, \ldots\}$.

   Using Bayes' Rule,

$$P(X_1 = k \mid X_3 = n) = \frac{P(X_1 = k, X_3 = n)}{P(X_3 = n)}$$

   For the event $\{X_1 = k, X_3 = n\}$ to occur, we need the following sequence to occur:

   (a) $k-1$ failures, followed by a success;

   (b) exactly 1 success somewhere in trials $k+1$ through $n-1$ (there are $n-1-k$ trials in this interval);

   (c) success on trial n.

   Thus, this probability is

$$\begin{aligned}
\frac{P(X_1 = k, X_3 = n)}{P(X_3 = n)} &= \frac{q^{k-1}p \cdot \binom{n-1-k}{1} p q^{n-2-k} \cdot p}{\binom{n-1}{2} q^{n-3} p^3} \\
&= \frac{\binom{n-1-k}{1} q^{n-3} p^3}{\binom{n-1}{2} q^{n-3} p^3} \\
&= \frac{n-1-k}{\binom{n-1}{2}}.
\end{aligned}$$

   Observe that this result does not depend on p! There is an intuitive explanation: knowing the 3rd success happened on n, we know there are two successes that must be placed somewhere amongst the first $n-1$ trials but do not know their locations. Since each trial had the same probability of success, these two successes are uniformly distributed amongst the $n-1$ trials. Then we are looking for where the first one occurs; this is like finding the number of cards until we draw our first ace.

3. In a room of n people, we are interested in finding out how many people share the same birthday. Assume as before that birthdays are independent and uniformly distributed across the year. Let N represent the number of people who share a birthday with at least one other person. Calculate $\mathbb{E}(N)$ and $Var(N)$.

For this problem, we will use indicators. Let I_j be the indicator variable that person j shares their birthday with at least one other person. Then, for all j, we note that

$$\begin{aligned}\mathbb{E}(I_j) &= P(I_j = 1)\\ &= 1 - P(\text{no one else has person } j\text{'s birthday})\\ &= 1 - \left(\frac{364}{365}\right)^{n-1}\end{aligned}$$

We use these indicators so that random variable N can then be expressed as a sum of the indicators: $N = I_1 + I_2 + \ldots + I_n$. By linearity of expectations,

$$\begin{aligned}\mathbb{E}(N) = \mathbb{E}\left(\sum_{j=1}^{n} I_j\right) &= \sum_{j=1}^{n} \mathbb{E}(I_j)\\ &= n\left(1 - \left(\frac{364}{365}\right)^{n-1}\right)\end{aligned}$$

This method works regardless of the dependence structure between the indicators. In fact, the indicators are not independent, which will matter for finding the variance.

To find $Var(N)$, we also write N as a sum of indicators, as we have seen before. Since the indicators are not independent, we will be concerned about $\mathbb{E}(I_j I_k)$. Observe this expectation will be the same for all pairs where $j \neq k$.

$$
\begin{aligned}
Var(N) &= \mathbb{E}(N^2) - [\mathbb{E}(N)]^2, \\
\text{where } \ \mathbb{E}(N^2) &= \mathbb{E}\left[\left(\sum_{j=1}^{n} I_j\right)^2\right] \\
&= \mathbb{E}\left(\sum_{j=1}^{n} I_j^2 + \sum_{j\neq k} I_j I_k\right) \\
&= n\mathbb{E}(I_1) + n(n-1)\mathbb{E}(I_1 I_2)
\end{aligned}
$$

To find $\mathbb{E}(I_1 I_2)$, we must find the probability that both person 1 and 2 share their birthday with at least one other person. Let A_1, A_2 represent these events. We proceed using DeMorgan's Rule, and the Inclusion-Exclusion rule:

$$
\begin{aligned}
\mathbb{E}(I_1 I_2) &= P(A_1 A_2) \\
&= 1 - P(A_1^c \cup A_2^c) \\
&= 1 - (P(A_1^c) + P(A_2^c) - P(A_1^c A_2^c)) \\
&= 1 - \left(2\left(\frac{364}{365}\right)^{n-1} - \left(\frac{364}{365}\right)\left(\frac{363}{365}\right)^{n-2}\right)
\end{aligned}
$$

This last term $(P(A_1^c A_2^c))$ is obtained by considering that persons 1 and 2 must have different birthdays, and all $n-2$ remaining people must have different birthdays from both of them.

Finally, putting it all together, we get

$$
\begin{aligned}
Var(N) = n&\left(1 - \left(\frac{364}{365}\right)^{n-1}\right) \\
&+ n(n-1)\left(1 - \left(2\left(\frac{364}{365}\right)^{n-1} - \left(\frac{364}{365}\right)\left(\frac{363}{365}\right)^{n-2}\right)\right) \\
&- \left(n\left(1 - \left(\frac{364}{365}\right)^{n-1}\right)\right)^2
\end{aligned}
$$

4. Let X be a Geometric (p) random variable on $\{1, 2, 3, \ldots\}$.

(a) Evaluate $\mathbb{E}(X^{-1})$. Hint: recall the Taylor series, $\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.

Recall the function rule for expectations:

$$\mathbb{E}(g(X)) = \sum_{\text{all } x} g(x)P(X = x)$$

Let $q = 1 - p$. Applying the function rule, we have:

$$\begin{aligned}
\mathbb{E}(X^{-1}) &= \sum_{k=1}^{\infty} k^{-1}P(X = k) \\
&= \sum_{k=1}^{\infty} k^{-1}q^{k-1}p \\
&= \frac{p}{q}\sum_{k=1}^{\infty}\frac{q^k}{k} \quad \text{(factoring out common terms)} \\
&= -\frac{p}{q}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(-q)^k}{k} \quad \text{(multiplying by } (-1)^{2k+2} = 1\text{)} \\
&= -\frac{p}{q}\log(1-q) \quad \text{(by Taylor series formula)} \\
&= -\frac{p}{q}\log(p).
\end{aligned}$$

(b) For p close to 1, what is this value, approximately? Your answer should be a very simple expression in terms of p.

Recall the approximation we have used before in this class: $\log(1+x) \approx x$ for small x. (This is obtained by using the first term of the Taylor series shown above as an approximation.)

For p close to 1, it is also true that q is small. Thus,

$$\begin{aligned}
\mathbb{E}(X^{-1}) &= -\frac{p}{q}\log(p) \\
&= -\frac{p}{q}\log(1-q) \\
&\approx -\frac{p}{q}(-q) = p.
\end{aligned}$$

Note: This example demonstrates in part (a) that for most functions $g(\cdot)$, $\mathbb{E}(g(X)) \neq g(\mathbb{E}(X))$. Yet, in part (b), we find that the two are approximately equal **in some cases**. Do not try to use this result for other types of problems!

5. Suppose there are two kinds of lottery tickets A and B. You buy A once a day and B once a week. You have a 1/25 chance of winning A if you buy it, and a 1/4 chance of winning B. You really want to win, so you keep buying both of them for a year (that is, 365 A tickets and 52 B tickets in total). Let X be the number of winning tickets you have in total. Find:

   (a) $\mathbb{E}(X)$.

   Let X_1 denote the number of daily winning tickets and X_2 the number of weekly winning tickets in a year. X_1 follows Bin(365, $\frac{1}{25}$), and X_2 follows Bin(52, $\frac{1}{4}$).
   $\mathbb{E}(X_1) = 365 \times \frac{1}{25} = \frac{365}{25} = \frac{73}{5} = 14.6$
   $\mathbb{E}(X_2) = 52 \times \frac{1}{4} = \frac{52}{4} = 13$
   Since $X = X_1 + X_2$, $\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 14.6 + 13 = 27.6$.

   (b) $Var(X)$.

   Using the fact that X_1 and X_2 are both binomial random variables, we can compute $Var(X_1)$ and $Var(X_2)$ easily.
   $Var(X_1) = 365 \times \frac{1}{25} \times \frac{24}{25} = \frac{24\times 365}{25\times 25} = \frac{1752}{125} = 14.016$
   $Var(X_2) = 52 \times \frac{1}{4} \times \frac{3}{4} = \frac{3\times 52}{4\times 4} = \frac{39}{4} = 9.75$
   Using the fact that X_1 and X_2 are independent, we can compute $Var(X)$ as the sum of $Var(X_1)$ and $Var(X_2)$. So $Var(X) = Var(X_1) + Var(X_2) = 14.016 + 9.75 = 23.766$

   (c) Approximately $P(X = 18)$.

   Let's start by approximating X_1 and X_2 seperately using **normal** distribution. The reason we're using normal distribution is because both $SD(X_1)$ and $SD(X_2)$ are greater than 3.
   Now X_1 follows $\mathcal{N}(14.6, 14.016)$ and X_2 follows $\mathcal{N}(13, 9.75)$.
   If we add two independent normal random variables, we get another normal random variable. So we have X following $\mathcal{N}(27.6, 23.766)$
   To compute $P(X = 18)$, we use continuity correction. So

$$
\begin{aligned}
P(X = 18) &= P(17.5 \leq X \leq 18.5) \\
&= P(X \leq 18.5) - P(X \leq 17.5) \\
&= P(\frac{X - 27.6}{\sqrt{23.766}} \leq \frac{18.5 - 27.6}{\sqrt{23.766}}) - P(\frac{X - 27.6}{\sqrt{23.766}} \leq \frac{17.5 - 27.6}{\sqrt{23.766}}) \\
&= P(Z \leq \frac{18.5 - 27.6}{\sqrt{23.766}}) - P(Z \leq \frac{17.5 - 27.6}{\sqrt{23.766}}) \\
&= \Phi(\frac{18.5 - 27.6}{\sqrt{23.766}}) - \Phi(\frac{17.5 - 27.6}{\sqrt{23.766}})
\end{aligned}
$$