Statistics 134 - Instructor: Adam Lucas

Final Exam Solutions

Tuesday, May 14, 2019

Print your name: ______________________________

SID Number: ______________________________

Exam Information and Instructions:

- You will have 170 minutes to take this exam. Closed book/notes/etc. No calculator or computer.
- We will be using Gradescope to grade this exam. Write any work you want graded on the front of each page, in the space below each question. Additionally, write your SID number in the top right corner on every page.
- Please use a dark pencil (mechanical or #2), and bring an eraser. *If you use a pen and make mistakes, you might run out of space to write in your answer.*
- Provide calculations or brief reasoning in every answer.
- Unless stated otherwise, you may leave answers as unsimplified numerical and algebraic expressions, and in terms of the Normal c.d.f. Φ. Finite sums are fine, but simplify any infinite sums.
- Do your own unaided work. Answer the questions on your own. The students around you have different exams.

I certify that all materials in the enclosed exam are my own original work.

Sign your name: ______________________________

GOOD LUCK!

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1. Let X, Y have joint density given by

$$f_{X,Y}(x,y) = \frac{\lambda}{y}e^{-\lambda y}, \quad 0 < x < y.$$

(a) Find the marginal distribution of Y.

$$\begin{aligned} f_Y(y) &= \int_0^y f_{X,Y}(x,y)dx \\ &= \int_0^y \frac{\lambda}{y}e^{-\lambda y}dx \\ &= \left[\frac{\lambda e^{-\lambda y}x}{y}\right]_{x=0}^{y} \\ &= \lambda e^{-\lambda y}, y > 0 \end{aligned}$$

Therefore, $Y \sim$ Exp (λ).

(b) Find the conditional density of X, given $Y = y$. You should recognize this as one of our named distributions.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{\frac{\lambda}{y}e^{-\lambda y}}{\lambda e^{-\lambda y}} \\ &= \frac{1}{y}, 0 < x < y. \end{aligned}$$

Therefore, $X|Y = y$ follows the Uniform $(0, y)$ distribution.

(c) Use (a) and (b) to find $\mathbb{E}(X)$.

This part relies on us obtaining the correct answers for part (a) and part (b). We should recognize $X \sim$ Unif $(0, Y)$, where $Y \sim$ Exp (λ). We then proceed using the law of iterated expectations:

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}(\mathbb{E}(X|Y)) \\ &= \mathbb{E}\left(\frac{Y}{2}\right) \\ &= \frac{1}{2\lambda}. \end{aligned}$$

2. Brian and Yiming play a game, where Brian and Yiming each draws five numbers without replacement from separate sets $\{1, 2, \ldots, 100\}$.

   (a) What is the expected number of common numbers in their choices?

   Let X be the number of common numbers in their choice.

   Define the indicator variable I_k to be the indicator that both Brian and Yiming have chosen the number k.

   $E(I_k) = \frac{5}{100}\frac{5}{100} = \frac{1}{400} \forall k \implies E(X) = \sum_{k=1}^{100} E(I_k) = 100 \times \frac{1}{400} = \frac{1}{4}$.

   Alternatively, students may write X as a sum of 5 indicators where the 2nd indicator is 1 if Brian's 2nd choice matches one of Yiming's numbers (this has probabilty 5/100). The expectation of X is then 5(5/100)=1/4.

   (b) What is the variance of the number of common numbers in their choices?

   $Var(X) = E(X^2) - [E(X)]^2 = E(X) + 100 \times 99E(I_1 I_2) - [E(X)]^2$

   $E(I_1 I_2) = \frac{\binom{5}{2}}{\binom{100}{2}}\frac{\binom{5}{2}}{\binom{100}{2}} \implies Var(X) = \frac{1}{4} + 100 \times 99\frac{\binom{5}{2}}{\binom{100}{2}}\frac{\binom{5}{2}}{\binom{100}{2}} - (\frac{1}{4})^2$

   (c) What is the probability that Brian and Yiming select at least one number in common?

   Seeing the numbers chosen by Brian as red balls and the other numbers as white balls. This can be modeled as Yiming drawing 5 balls out of a box containing 5 red balls and 95 white balls, and X is the number of red balls Yiming draws out. It is directly seen that X has the hypergeometric distribution;

   $$P(X = x) = \frac{\binom{5}{x}\binom{95}{5-x}}{\binom{100}{5}}$$

   In particular, the probability that the choices of Brian and Yiming have at least one number in common is $1 - P(X = 0) = 1 - \frac{\binom{95}{5}}{\binom{100}{5}}$.

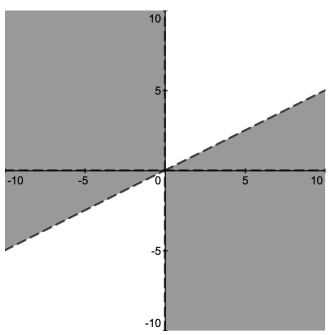
3. Let X, Y be independent standard normal.

   (a) Find $P(X < kY)$.

   We begin by observing that $X < kY$ is equivalent to $X - kY < 0$. Then, since $W = X - kY$ is a linear combination of independent normals, $W \sim \mathcal{N}(0, 1 + k^2)$. Thus the desired probability is

$$P(X < kY) = P(X - kY < 0) = \frac{1}{2}$$

   (b) It can be shown that $Z = Y/X$ follows the Cauchy distribution. Use rotational symmetry of X, Y to find $P(Z < k)$, for $k > 0$, *without* directly using the CDF of the Cauchy distribution given on your reference sheet. Hint: the region $Y/X < k$ is shaded in the graph below, for $k = 0.5$.



   We use rotational symmetry to reduce the problem to finding the angles corresponding to these regions, divided by 2π. Thus the desired probability is

$$\begin{aligned} P(Y/X < k) &= P(X < 0, Y > 0) + P(X > 0, Y < 0) + P(0 < Y < kX) + P(kX < Y < 0) \\ &= \frac{1}{4} + \frac{1}{4} + 2 \cdot \frac{\arctan(k)}{2\pi} \\ &= \frac{1}{2} + \frac{\arctan(k)}{\pi} \end{aligned}$$

   This formula also holds for $k < 0$, but the region of interest looks slightly different. It is consistent with the CDF for a Cauchy that can be obtained directly by integrating the Cauchy PDF.

4. Consider $X \sim$ Beta (r, s). We have not yet seen the MGF of a Beta; let us now derive the result. This requires a slightly different approach than we have seen before, as the result is not so elegant as other MGFs we have seen.

   (a) Calculate $E(X^k)$, where k is a positive integer. You may leave your result in terms of $\Gamma(\cdot)$.
   Using the function rule,

$$
\begin{aligned}
E(X^k) &= \int_0^1 x^k f_X(x)dx \\
&= \int_0^1 \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r+k-1}(1-x)^{s-1}dx \\
&= \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \cdot \frac{\Gamma(r+k)\Gamma(s)}{\Gamma(r+s+k)} \int_0^1 \frac{\Gamma(r+s+k)}{\Gamma(r+k)\Gamma(s)} x^{r+k-1}(1-x)^{s-1}dx \\
&= \frac{\Gamma(r+s)\Gamma(r+k)}{\Gamma(r)\Gamma(r+s+k)}.
\end{aligned}
$$

   (b) Let $b_k = E(X^k)$ denote the result you have calculated above. Use the Taylor series for e^x to write $M_X(t)$ as a (infinite) summation containing these b_k terms. Do not attempt to simplify this result!
   Recall that $M_X(t) = E(e^{tX})$. Using the Taylor expansion, we calculate:

$$
\begin{aligned}
M_X(t) &= E\left(\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\right) \\
&= \sum_{k=0}^{\infty} \frac{t^k E(X^k)}{k!} \\
&= \sum_{k=0}^{\infty} \frac{t^k b_k}{k!}
\end{aligned}
$$

   (c) Now we must check, is this MGF defined? Give simple lower and upper bounds for all the b_k terms. Use this to give lower and upper bounds for $M_X(t)$, for $t > 0$. (Hint: think about the possible values of X^k. Your bounds should not depend on k.)
   Observe that $X^k \in (0, 1)$ for all k. So $0 < E(X^k) < 1$ for all k. Thus, for $t > 0$,

$$
\begin{aligned}
0 < M_X(t) &= \sum_{k=0}^{\infty} \frac{t^k b_k}{k!} \\
&< \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t
\end{aligned}
$$

   So $0 < M_X(t) < e^t$ for all $t > 0$. This series indeed converges, using the comparison test for series.

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5. Let $X_1, X_2, \ldots, X_n$ be independent Poisson random variables with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $Y = \sum_{i=1}^n X_i$. Find the distribution of $X_1 \mid Y = k$ and determine its expectation.

   In brief, $X_i \mid Y = k \sim \text{Bin}(k, \lambda_1 / \sum_{j=1}^n \lambda_j)$.

   The argument goes as follows. Let $\mu = \sum_{i=1}^n \lambda_1$ and let $Z = \sum_{i=2}^n X_i$, which we know to be Poisson with parameter $\mu - \lambda_1$. Then,

$$\begin{aligned}
\mathbb{P}(X_1 = a \mid Y = k) &= \frac{\mathbb{P}(X_1 = a \cap Y = k)}{\mathbb{P}(Y = k)} \\
&= \frac{\mathbb{P}(X_1 = a) \cdot \mathbb{P}(Z = k - a)}{\mathbb{P}(Y = k)} \\
&= \frac{\lambda_1^a e^{-\lambda_1}}{a!} \frac{(\mu - \lambda_1)^{k-a} e^{-(\mu - \lambda_1)}}{(k-a)!} \frac{k!}{\mu^k e^{-\mu}} \\
&= \binom{k}{a} \left(\frac{\lambda_1}{\mu}\right)^a \left(\frac{\mu - \lambda_1}{\mu}\right)^{k-a}.
\end{aligned}$$

   Since $X_1 \mid Y = k$ is Binomial, its expectation is $k\lambda_1 / \sum_{j=1}^n \lambda_j$.

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6. (a) State Markov's inequality.

For $X \geq 0$ and any $a > 0$, $\mathbb{P}(X \geq a) \leq a^{-1}\mathbb{E}X$.

(b) Let Y be a random variable with moment-generating function M_Y. Use Markov's inequality to prove that, for any $t > 0$,

$$\mathbb{P}(Y \geq a) \leq e^{-t \cdot a} M_Y(t).$$

While Y may not be nonnegative, $e^{t \cdot Y}$ certainly is. We may therefore write $\mathbb{P}(Y \geq a) = \mathbb{P}(e^{t \cdot Y} \geq e^{t \cdot a})$ and then apply Markov's inequality to obtain

$$\mathbb{P}(Y \geq a) \leq e^{-t \cdot a}\mathbb{E}\left[e^{t \cdot Y}\right] = e^{-t \cdot a} M_Y(t).$$

(c) Let X be a continuous, nonnegative random variable such that $E(X) = \mu$. Consider its median m_X; recall that the median of X is the value such that $P(X \leq m_X) = P(X \geq m_X)$. In terms of μ, what is the largest possible value of m_X?

This problem requires a clever application of Markov's Inequality. Observe that $1/2 = P(X \geq m_X)$, and $P(X \geq m_X) \leq \mu/m_X$, by Markov. Rearranging,

$$0.5 \leq \frac{\mu}{m_X} \iff m_X \leq 2\mu.$$

7. (a) A family is getting ready for their trip to Yosemite. Each person is in their room, packing their bags. For each person, the time it takes them to pack their bag is exponentially distributed and independent of the time it takes any other person. On average, it takes each parent 1 hour and each child 2 hours to get ready. In a family with 2 parents and 4 children, what is the probability that it takes the family more than 2 hours to get ready?

Solution: We are asked to compute the probability that $M = \max(X_1, X_2, X_3, X_4, X_5, X_6)$ where X_1, X_2 are exp(1) and X_3, X_4, X_5, X_6 are exp(1/2), is at most 2. We have that

$$\mathbb{P}(M \leq x) = \mathbb{P}(X_1, \ldots, X_6 \leq x) = \mathbb{P}(X_1 \leq x) \cdots \mathbb{P}(X_6 < x) = (1 - e^{-x})^2 \left(1 - \frac{1}{2} e^{-\frac{x}{2}}\right)^4.$$

So the probability that it takes them more than 2 hours is

$$\mathbb{P}(M > 2) = 1 - \mathbb{P}(M \leq 2) = 1 - (1 - e^{-2})^2 \left(1 - \frac{1}{2} e^{-1}\right)^4$$

.

(b) After setting up the tents the family wants to rent bicycles. The rental place does only have one bicycle when they get there. Assume that the bikes being returned follow a Poisson process with rate $\frac{1}{10}$ and that nobody else is waiting for bikes. Give two expressions for the probability that the family will have to wait more than 30 minutes before they can start their biking tour. (No need to simplify!)

If B_t is the number of bikes returned by time t this question is asking for the probability that B_{30} is smaller or equal to 4. B_{30} is Poisson $30\frac{1}{10} = 3$. Thus

$$\mathbb{P}(B_{30} \leq 4) = \sum_{k=0}^{4} e^{-3} \frac{3^k}{k!}.$$

Alternatively, with T_i denoting the arrival time of the ith bike, we have that T_i is $gamma(i, \frac{1}{10})$, so $\mathbb{P}(B_{30} \leq 4) = \mathbb{P}(T_5 > 30) = \int_{30}^{\infty} \frac{x^4 e^{-\frac{x}{10}}}{4! 10^5}$

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8. Let X, Y be random variables such that for $x, y \in \{1, 2, 3\}$,

$$\mathbb{P}(X = x, Y = y) = \frac{x+y}{36}.$$

(a) Give an expression for $\mathbb{P}(X = Y)$.

$\mathbb{P}(X = Y) = \sum_{k=1}^{3} \mathbb{P}((X = k, Y = k)) = \frac{2+4+6}{36} = \frac{12}{36} = \frac{1}{3}$.

(b) Using (a), find an expression for $\mathbb{P}(X < Y)$. (Hint: Use symmetry.)

Since X and Y are identically distributed, $\mathbb{P}(Y < X) = \mathbb{P}(X < Y)$. So $2\mathbb{P}(X < Y) = 1 - \frac{1}{3}$ and $\mathbb{P}(X < Y) = \frac{1}{3}$.

(c) Are X and Y independent?

No: $\mathbb{P}(X = 1, Y = 1) = \frac{2}{36} = \frac{1}{18}$, but

$$\mathbb{P}(X = 1) = \mathbb{P}(Y = 1) = \sum_{k=1}^{3} \frac{1+k}{36} = \frac{3}{36} + \frac{6}{36} = \frac{9}{36} = \frac{1}{4}$$

so

$$\mathbb{P}(X = 1)\mathbb{P}(Y = 1) = \frac{1}{16} \neq \frac{1}{18}.$$

(d) What is the probability mass function of $Z = X + Y$?

The range of Z is $\{2, 3, \dots, 6\}$. This is an application of the convolution formula $P(Z = s) = \sum_{x=1}^{s} P(X = x, Y = s - x)$ or the student can notice that the joint pmf only depends on the sum of X and Y.

$$\mathbb{P}(Z = 2) = \frac{2}{36}$$
$$\mathbb{P}(Z = 3) = 2 \cdot \frac{3}{36}$$
$$\mathbb{P}(Z = 4) = 3 \cdot \frac{4}{36}$$
$$\mathbb{P}(Z = 5) = 2 \cdot \frac{5}{36}$$
$$\mathbb{P}(Z = 6) = \frac{6}{36}$$

9. Let $X \sim$ Uniform $(-1, 1)$ (this is a continuous uniform random variable).

(a) Compute the density of $Y = e^X$.

Solution: For $y > 0$, $y = e^x \Leftrightarrow x = \ln(y)$ and $\frac{dg(x)}{dx} = e^x$. By the change of variables formula

$$f_Y(y) = \frac{f_X(\ln(y))}{e^{\ln(y)}} = \frac{1}{2y} \text{ for } e^{-1} < y < e,$$

since $-1 < \ln(y) < 1 \Leftrightarrow e^{-1} < y < e$.

(b) Let now X_1, X_2 be i.i.d. uniform random variables, and for each $i = 1, 2$, let $Y_i = e^{X_i}$. What is the joint density of $Y_{(1)}$ and $Y_{(2)}$, the minimum and the maximum of the Y_i's?

Thus for $e^{-1} < x < y < e$,

$$f_{Y_{(1)},Y_{(2)}}(x, y) = \binom{2}{1} \frac{1}{2x} \frac{1}{2y} = \frac{1}{2} \frac{1}{xy}.$$

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10. Let X and Y be the heights (in centimeters) of mothers and daughters in a family. We know that

$$(X, Y) \sim BN(\mu_X = 170, \mu_Y = 170, \sigma_X^2 = 5^2, \sigma_Y^2 = 5^2, \rho = .8),$$

where BN stands for Bivariate Normal.

(a) What is the distribution of $Y - X$? Explain fully with parameters.

(b) Find the probability that the daughter is more than 7 cm taller than her mother.

(c) Consider X_1, X_2 be the heights of two mothers selected at random from the population (so X_1 and X_2 are independent). Let $V = X_1 - X_2$ and $W = 3X_1 - 2X_2$. Find $Corr(V, W)$.

Solution:

(a) Since (X,Y) is bivariate normal, Y-X is normal with mean $E(Y - X) = 0$ and variance $Var(Y - X) = Var(X) + Var(Y) - 2Cov(X, Y)$. Note that $Cov(X, Y) = Corr(X, Y)\sigma_X\sigma_Y = 0.8(5^2) = 20$. So $Y - X$ follows the Normal (0, 25 + 25 - 40) distribution.

(b) $P(Y > X + 7) = P(Y - X > 7)$ and by part (a), $P(Y - X > 7) = 1 - \Phi(\frac{7}{\sqrt{10}})$.

(c) $Cov(V, W) = Cov(X_1 - X_2, 3X_1 - 2X_2) = 3\sigma_X^2 + 2\sigma_X^2 = 3 \cdot 5^2 + 2 \cdot 5^2 = 125$
$Var(V) = Var(X_1 - X_2) = \sigma_X^2 + \sigma_X^2 = 25 + 25 = 50$
$Var(W) = Var(3X_1 - 2X_2) = 9\sigma_X^2 + 4\sigma_X^2 = 9 \cdot 25 + 4 \cdot 25 = 325$
$Corr(V, W) = \frac{Cov(V,W)}{SD(V)SD(W)} = 125/\sqrt{50 \cdot 325} = .98$