Stat 134: Review of Joint Densities

May 8th, 2019

Narrative

1. Densities are one of the ways we characterize *continuously*-valued rvs. For a continuously-valued rv X, $\mathbb{P}(X = x) = 0$ for all x. We therefore instead consider the probability $\mathbb{P}(X \in \mathrm{d}x)$.

2. We can more conveniently express these probabilities for every x by packaging them into a function—the density $f_X(x)$. In this way, we can write $\mathbb{P}(X \in \mathrm{d}x) = f_X(x)\mathrm{d}x$. Note that $\mathrm{d}x$ is an *abuse of notation* because, in the context of "$\mathbb{P}(X \in \mathrm{d}x)$", we mean $\mathrm{d}x$ to be a set like $\left(x - \frac{1}{2}\mathrm{d}x, x + \frac{1}{2}\mathrm{d}x\right)$ but, in the context of "$f_X(x)\mathrm{d}x$", we mean it as an infinitesimal length.

3. This idea also works for joint probabilities as $\mathbb{P}(X \in \mathrm{d}x, Y \in \mathrm{d}y) = f_{X,Y}(x,y)\mathrm{d}x\,\mathrm{d}y$. A joint density $f_{X,Y}$ may be integrated to obtain: (i) probabilities/expectations involving *functions* of X and Y; (ii) the marginal densities f_X and f_Y; and (iii) the joint or marginal distributions $F_{X,Y}$, F_X, and F_Y.

4. The joint density contains at least as much "information" as do the collection of marginals. This is because, while we can always obtain the marginals by integrating the joint density, we cannot get the joint density from the marginals, in general. A notable exception is when X and Y are independent, as then $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

Conceptual Review

Suppose X, Y are rvs with joint density $f_{X,Y}$ over the region $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$. Setup integrals for:

(i) $f_X(x)$;

(ii) $F_Y(y)$;

(iii) $\mathbb{P}(Y < X + 5)$;

(iv) $\mathbb{E}X$;

(v) $\mathbb{E}(g(X,Y))$.

Problem 1

Suppose X, Y are standard bivariate normal with correlation ρ. Find $f_{X,Y}$ in terms of ϕ.

Problem 2

Based on each of the joint densities below, *with minimal calculation*, identify the marginals of X and Y.

a. $f_{X,Y}(x,y) = 3360x^3(y-x)^2(1-y)$, $0 < x < y < 1$ (Bonus: how did we compute the constant of 3360)?

b. $f_{X,Y}(x,y) = \lambda^3 e^{-\lambda y}(y-x)$, $0 < x < y$ (Hint: $X \sim \text{Exp}(\lambda)$).

c. $f_{X,Y}(x,y) = e^{-4y}$, $0 < x < 4$, $0 < y$.

Problem 3

Let (X,Y) be a point chosen uniformly from $\{(x,y) : x > 0, y > 0, x^2 + y^2 < 4\}$. Let $R = \sqrt{X^2 + Y^2}$. Find:

a. $f_{X,Y}(x,y)$;

b. $f_R(r)$; and

c. $\mathbb{P}(cX > Y)$, for $c > 0$.

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