

**Statistics 134 - Instructor: Adam Lucas**

**Final Exam**

Wednesday, May 9, 2018

**Print your name:** \_\_\_\_\_

**SID Number:** \_\_\_\_\_

**Name of the person to your left:** \_\_\_\_\_

**Name of the person to your right:** \_\_\_\_\_

**Exam Information and Instructions:**

- You will have 170 minutes to take this exam. Closed book/notes/etc. No calculator or computer.
- We will be using Gradescope to grade this exam. Write any work you want graded on the front of each page, in the space below each question. Additionally, write your SID number in the top right corner on every page.
- Please use a dark pencil (mechanical or #2), and bring an eraser. *If you use a pen and make mistakes, you might run out of space to write in your answer.*
- Provide calculations or brief reasoning in every answer.
- Unless stated otherwise, you may leave answers as unsimplified numerical and algebraic expressions, and in terms of the Normal c.d.f.  $\Phi$ . Finite sums are fine, but simplify any infinite sums and integrals.
- Do your own unaided work. Answer the questions on your own. The students around you have different exams.

*I certify that all materials in the enclosed exam are my own original work.*

**Sign your name:** \_\_\_\_\_

GOOD LUCK!

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1. (12 pts) Yiming and Brian are playing darts. Suppose each dart's distance from the bullseye is a standard Rayleigh variable, independent of all other darts. If a dart is within 1 unit of the bullseye, 2 points are earned; if in 1 to 4 units of the bullseye, 1 point is earned; otherwise no points.

(a) Find the expected point value of each dart throw.

- (b) Now let  $p_i$  represent the probability of scoring  $i$  points. Yiming and Brian take turns throwing; Yiming throws first. In terms of  $p_0, p_1, p_2$ , what is the probability Yiming is first to score at least one point?

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- (c) (continued from Q1): Darts are thrown until one scores two points. Let  $X$  denote the number of points scored until stopping (not including the final throw). Find  $\mathbb{E}(X)$ .
2. (6 pts) Beloved Stat 134 students Jonny and Ethan have very different study habits for the final. Jonny quits studying at a constant instantaneous failure rate  $\lambda = \frac{1}{2}$  (i.e. average 2 hours). Ethan randomly quits anytime between 0 and 4 hours (also average 2 hours). You may assume Jonny and Ethan study independently. What is the chance the first quitter stops studying between 2 and 3 hours?

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3. (10 pts) Phone calls arrive into a telephone exchange according to a Poisson arrival process at rate  $\lambda$  per unit time. This exchange serves three regions, and an incoming call gets routed to region  $i$  with probability  $p_i$ , for  $i = 1, 2, 3$ . Note that  $p_1 + p_2 + p_3 = 1$ .

(a) Let  $N_t^i$  denote the numbers of calls routed to region  $i$  in time  $t$  starting from time 0. Find  $\mathbb{P}(N_t^1 = j, N_t^2 = k)$ .

(b) On a particular day, the telephone exchange malfunctions, and it fails to route each incoming call with probability  $q$ , independently of all other calls; such a “dropped” call is lost forever and it does not get transmitted to any region. On this unlucky day, what is the expected waiting time between the successive calls received by region 1?

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4. (6 pts) For each distribution of  $X$  indicated in the parts below, evaluate  $Var(X)$  as a simple fraction, without involving the constants  $c_1, c_2, c_3$ .

(a)  $X$  with values in  $[0, 1]$  and density  $f_X(x) = c_1x^2(1-x)^3$  for  $0 \leq x \leq 1$ .

(b)  $X$  with values in  $[0, \infty)$  and density  $c_2x^4e^{-3x}$  for  $x \geq 0$

(c)  $X$  with possible values  $0, 1, 2, \dots$  and  $\mathbb{P}(X = k) = \frac{c_3}{k!}$  for  $k = 0, 1, 2, \dots$

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5. (10 pts) The joint density of  $X$  and  $Y$  is

$$f(x, y) = \begin{cases} \frac{4y}{x} & \text{for } 0 < x < 1 \text{ and } 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

Find the following:

(a)  $\mathbb{E}(XY)$

(b) The marginal density of  $X$

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(c) (continued from Q5):  $\mathbb{E}(Y|X)$

6. (6 pts) Let  $U$  follow a Uniform  $(0, 1)$  distribution. Find the distribution of  $-\log(U)$ .

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7. (10 pts) Suppose  $X$  and  $Y$  are independent and normally distributed with mean 0 and variance 1. Find the density of  $X/Y$ .



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8. (12 pts) There are 10 students preparing for a TV show audition. Eight of them can sing while five of them can dance (i.e. 3 students can sing and dance). All selections are made without replacement, except in part (a).

(a) If a director randomly selects 7 students **with replacement**, what is the chance that she selects exactly 4 singers? Hint: a student is either a singer or not a singer.

(b) Two students are randomly selected **without replacement**. What is the chance that the first student can only sing given that the second student can dance?

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(c) A student has one of three possible skills:

- i. sing only
- ii. dance only
- iii. sing and dance

If 5 students are randomly selected, how many different skills do you expect the 5 students will have?

**For example:** if all 5 students can both sing and dance then the number of different skills is 1. If 2 students can sing and dance, two students can dance only and one student can sing only then the 5 students would have 3 different skills.

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9. (8 pts) People arrive at a store according to a Poisson process of rate  $\lambda = 2$  per minute. Let  $T_i$  denote the arrival time in minutes of the  $i^{\text{th}}$  customer.

(a) What is the expected value of the third arrival  $T_3$ ?

(b) What is the expected value of the third arrival, given no customers have arrived in 2 minutes?

(c) What is the probability that the third arrival  $T_3$  is more than 1 minute after  $T_1$ ?

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10. (12 pts) Suppose we have a collection of  $n$  i.i.d variables,  $X_1, X_2, \dots, X_n \sim \mathcal{N}(0, \sigma^2)$ . We are interested in the relationship between a single data point  $X_1$  and the sample mean,  $\bar{X} = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$ .

(a) Find  $Corr(X_1, \bar{X})$ . Use this to identify the joint distribution of  $(X_1, \bar{X})$ . Be sure to state all parameters. Write your answer as a simplified fraction.

(b) How large must  $X_1$  be such that there is a 95% chance that the sample mean is greater than zero?

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(c) (continued from Q10): Find  $\mathbb{P}(\bar{X} > 0 \mid X_1 > 0)$ . (Use the conditional probability rule.)