Stat 134 Spring 2018: Midterm Review Solutions

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1. Flip a coin 8 times. Given that there are 3 heads, find the probability that:

   (a) The heads are not consecutive.

   Since we're looking for a probability, let's figure out the denominator and numerator separately. The total number of ways to place 3 heads in 8 coin tosses is going to be $\binom{8}{3}$, where order does not matter. So, $\binom{8}{3}$ is our denominator.

   To make sure no head is next to another head, let's lay down the tails first. With 5 tails, 6 slots are created where one head can go in.

   Note that different from problem 2 on Feb. 21st worksheet, once a head goes into a slot, neither does it create an extra slot nor can other heads go into the same slot.

   So our answer is

$$\frac{\binom{6}{3}}{\binom{8}{3}}$$

   (b) The heads are all next to each other.

   Now if we all of the heads to be next to each other, we have to place all three of them into one slot. To do this, let's consider all three of them as a group. Now I just need to pick one slot to place this **one** group. So our numerator is $\binom{6}{1}$. Hence, our final answer is

$$\frac{\binom{6}{1}}{\binom{8}{3}}$$

   (c) There are 2 heads next to each other, and 1 other head by itself.

   Let's consider 2 heads as a group, and the one head left as another group. Now we need to pick 2 slots to place the two groups, and that gives us $\binom{6}{2}$. But keep in mind that we have to pick one slot from the 2 for the group of 2 heads to go in! The idea behind this is that the two groups are **not identical**, just like triples and pairs are not identical in a poker hand. So our numerator is $\binom{6}{2}\binom{2}{1}$. Our final answer is

$$\frac{\binom{6}{2}\binom{2}{1}}{\binom{8}{3}}$$

2. An urn contains $n+1=51$ marbles which are either black or white. Shake up the urn, and draw $n=50$ marbles leaving one marble in the urn. Given that there are $k=40$ black marbles from the first $n=50$ draws, find the probability that the last ball in the urn is black.

   Let's use X to denote the number of black balls in the n draws, B denote the last ball is black and W denote the last ball is white.

   There are two possibilities for the $n+1=51$ balls in the urn, and let's denote the two theories we have T_1 and T_2. Without loss of generality, let's make T_1 represent the theory where there are 41 black balls and 10 white balls, and T_2 represent the theory that there are 40 black balls and 11 white balls. The two theories are **equally likely** so $P(T_1)=P(T_2)=0.5$.

   Now let's figure out the conditional probability that the color of the last ball given T_1 and T_2 respectively.

   For T_1, since there are 41 black balls and 10 white balls, the probability of last ball being black is the same as the first draw being black, which is the number of black balls over the total number of balls. So, $P(B|T_1)=\frac{41}{51}$, and $P(W|T_1)=\frac{10}{51}$.

   For T_2, by similar arguments, $P(B|T_2)=\frac{40}{51}$, and $P(W|T_2)=\frac{11}{51}$.

   The probability we're looking for is

$$
\begin{aligned}
P(B|X=k) &= \frac{P(X=k,B)}{P(X=k)} \\
&= \frac{P(T_1)\times P(B|T_1)}{P(T_1)\times P(B|T_1)+P(T_2)\times P(W|T_2)} \\
&= \frac{\frac{1}{2}\times\frac{41}{51}}{\frac{1}{2}\times\frac{41}{51}+\frac{1}{2}\times\frac{11}{51}} \\
&= \frac{41}{52}
\end{aligned}
$$

3. In a room of n people, we are interested in finding out how many people share the same birthday. Assume as before that birthdays are independent and uniformly distributed across the year. Let N represent the number of people who share a birthday with at least one other person. Calculate $\mathbb{E}(N)$.

For this problem, we will use indicators. Let I_j be the indicator variable that person j shares their birthday with at least one other person. Then, for all j, we note that

$$\begin{aligned}\mathbb{E}(I_j) &= P(I_j = 1)\\ &= 1 - P(\text{no one else has person } j\text{'s birthday})\\ &= 1 - \left(\frac{364}{365}\right)^{n-1}\end{aligned}$$

We use these indicators so that random variable N can then be expressed as a sum of the indicators: $N = I_1 + I_2 + \ldots + I_n$. By linearity of expectations,

$$\begin{aligned}\mathbb{E}(N) = \mathbb{E}\left(\sum_{j=1}^{n} I_j\right) &= \sum_{j=1}^{n} \mathbb{E}(I_j)\\ &= n\left(1 - \left(\frac{364}{365}\right)^{n-1}\right)\end{aligned}$$

This works regardless of the dependency relationships between I_i, I_j, because we are taking the expectation of a linear combination (a sum) of the indicators.

4. Let X be a Geometric (p) random variable on $\{1, 2, 3, \ldots\}$.

(a) Evaluate $\mathbb{E}(X^{-1})$. Hint: recall the Taylor series, $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$.

Recall the function rule for expectations:

$$\mathbb{E}(g(X)) = \sum_{\text{all } x} g(x) P(X = x)$$

Let $q = 1 - p$. Applying the function rule, we have:

$$\begin{aligned}
\mathbb{E}(X^{-1}) &= \sum_{k=1}^{\infty} k^{-1} P(X = k) \\
&= \sum_{k=1}^{\infty} k^{-1} q^{k-1} p \\
&= \frac{p}{q} \sum_{k=1}^{\infty} \frac{q^k}{k} \quad \text{(factoring out common terms)} \\
&= -\frac{p}{q} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-q)^k}{k} \quad \text{(multiplying by } (-1)^{2k+2} = 1) \\
&= -\frac{p}{q} \log(1-q) \quad \text{(by Taylor series formula)} \\
&= -\frac{p}{q} \log(p).
\end{aligned}$$

(b) For p close to 1, what is this value, approximately? Your answer should be a very simple expression in terms of p.

Recall the approximation we have used before in this class: $\log(1+x) \approx x$ for small x. (This is obtained by using the first term of the Taylor series shown above as an approximation.)

For p close to 1, it is also true that q is small. Thus,

$$\begin{aligned}
\mathbb{E}(X^{-1}) &= -\frac{p}{q} \log(p) \\
&= -\frac{p}{q} \log(1-q) \\
&\approx -\frac{p}{q}(-q) = p.
\end{aligned}$$

Note: This example demonstrates in part (a) that for most functions $g(\cdot)$, $\mathbb{E}(g(X)) \neq g(\mathbb{E}(X))$. Yet, in part (b), we find that the two are approximately equal **in some cases**. Do not try to use this result for other types of problems!

5. Suppose there are two kinds of lottery tickets A and B. You buy A once a day and B once a week. You have a 1/25 chance of winning A if you buy it, and a 1/4 chance of winning B. You really want to win, so you keep buying both of them for a year (that is, 365 A tickets and 52 B tickets in total). Let X be the number of winning tickets you have in total. Find:

(a) $\mathbb{E}(X)$.

Let X_1 denote the number of daily winning tickets and X_2 the number of weekly winning tickets in a year. X_1 follows Bin(365, $\frac{1}{25}$), and X_2 follows Bin(52, $\frac{1}{4}$).

$\mathbb{E}(X_1) = 365 \times \frac{1}{25} = \frac{365}{25} = \frac{73}{5} = 14.6$

$\mathbb{E}(X_2) = 52 \times \frac{1}{4} = \frac{52}{4} = 13$

Since $X = X_1 + X_2$, $\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) = 14.6 + 13 = 27.6$.

(b) $Var(X)$.

Using the fact that X_1 and X_2 are both binomial random variables, we can compute $Var(X_1)$ and $Var(X_2)$ easily.

$Var(X_1) = 365 \times \frac{1}{25} \times \frac{24}{25} = \frac{24\times365}{25\times25} = \frac{1752}{125} = 14.016$

$Var(X_2) = 52 \times \frac{1}{4} \times \frac{3}{4} = \frac{3\times52}{4\times4} = \frac{39}{4} = 9.75$

Using the fact that X_1 and X_2 are independent, we can compute $Var(X)$ as the sum of $Var(X_1)$ and $Var(X_2)$. So $Var(X) = Var(X_1) + Var(X_2) = 14.016 + 9.75 = 23.766$

(c) Approximately $P(X = 18)$.

Let's start by approximating X_1 and X_2 seperately using **normal** distribution. The reason we're using normal distribution is because both $SD(X_1)$ and $SD(X_2)$ are greater than 3.

Now X_1 follows $\mathcal{N}(14.6, 14.016)$ and X_2 follows $\mathcal{N}(13, 9.75)$.

If we add two independent normal random variables, we get another normal random variable. So we have X following $\mathcal{N}(27.6, 23.766)$

To compute $P(X = 18)$, we use continuity correction. So

$$\begin{aligned}
P(X = 18) &= P(17.5 \leq X \leq 18.5) \\
&= P(X \leq 18.5) - P(X \leq 17.5) \\
&= P(\frac{X - 27.6}{\sqrt{23.766}} \leq \frac{18.5 - 27.6}{\sqrt{23.766}}) - P(\frac{X - 27.6}{\sqrt{23.766}} \leq \frac{17.5 - 27.6}{\sqrt{23.766}}) \\
&= P(Z \leq \frac{18.5 - 27.6}{\sqrt{23.766}}) - P(Z \leq \frac{17.5 - 27.6}{\sqrt{23.766}}) \\
&= \Phi(\frac{18.5 - 27.6}{\sqrt{23.766}}) - \Phi(\frac{17.5 - 27.6}{\sqrt{23.766}})
\end{aligned}$$