SOLUTIONS TO THE COVARIANCE REVIEW

Conceptual Review.

(a) For every n random variables $X_1, \ldots, X_n$ we have:

$$\mathrm{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \mathrm{Cov}(X_i, X_j)$$

(b) If (X, Y) is a standard bivariate normal with correlation ρ, then if we define:

$$Z := \frac{Y - \rho X}{\sqrt{1-\rho^2}}$$

then as $\mathrm{Cov}(Z, X) = 0$, the random variables Z and X are independent, and furthermore as $\mathbb{E}[Z] = 0$ and $\mathrm{Var}(Z) = 1$, then Z is also a standard normal random variable. Hence we have the following equalities in distribution:

$$\begin{aligned}(Y|X = x) &\overset{d}{=} (\rho X + \sqrt{1-\rho^2} Z | X = x) \\ &\overset{d}{=} (\rho x + \sqrt{1-\rho^2} Z | X = x) \\ &\overset{d}{=} (\rho x + \sqrt{1-\rho^2} Z) \\ &\overset{d}{=} \mathcal{N}(\rho x, 1-\rho^2)\end{aligned}$$

By symmetry we then also have:

$$(X|Y = y) \overset{d}{=} (\rho y, 1-\rho^2)$$

Problem 1. The arrival times of a Poisson process of rate λ can be written as :

$$T_r = \sum_{k=1}^{r} X_k$$

where X_k is the waiting time between the $(k-1)$-th arrival and the k-th arrival. The random variables $X_1, \ldots, X_k$ are i.i.d. with distribution $\mathrm{Exp}(\lambda)$. Hence:

$$\mathrm{Cov}(T_1, T_3) = \mathrm{Cov}(X_1, X_1 + X_2 + X_3) = \mathrm{Var}(X_1)$$

On the other hand, by using part (a) of the conceptual review we get:

$$\mathrm{Var}(T_3) = \mathrm{Var}(X_1 + X_2 + X_3) = 3\mathrm{Var}(X_1)$$

Hence:

$$\mathrm{Corr}(T_1, T_3) = \frac{\mathrm{Cov}(T_1, T_3)}{\sqrt{\mathrm{Var}(T_1)\mathrm{Var}(T_3)}} = \frac{1}{\sqrt{3}}$$

Problem 2. Similarly as the first problem we can write :

$$W_r = \sum_{k=1}^{r} X_k$$

where the X_k's are the number of tosses between the $(k-1)$-th Head and the k-th Head. They are all i.i.d's with common distribution Geom(p). Now:

$$\mathrm{Cov}(W_1, W_r) = \mathrm{Cov}(X_1, X_1 + \cdots + X_r) = \mathrm{Var}(X_1)$$

and:

$$\mathrm{Var}(W_r) = \mathrm{Var}(\sum_{k=1}^{r} X_k) = r\mathrm{Var}(X_1)$$

Hence:

$$\mathrm{Corr}(W_1, W_r) = \frac{1}{\sqrt{r}}$$

Problem 3. Let us first find the pdf of $\max(X, Y)$ we have for all $x \in \mathbb{R}$:

$$\begin{aligned}
\mathbb{P}[\max(X,Y) \in dx] &= \mathbb{P}[X \in dx, Y < x] + \mathbb{P}[Y \in dx, X < x] \\
&= 2\mathbb{P}[X \in dx, Y < x] \\
&= 2\mathbb{P}[X \in dx]\mathbb{P}[Y < x | X = x] \\
&= 2\mathbb{P}[X \in dx]\mathbb{P}[\rho x + \sqrt{1-\rho^2} Z < x] \\
&= 2\phi(x)dx\mathbb{P}[Z < \frac{(1-\rho)}{\sqrt{1-\rho^2}} x] \\
&= 2\phi(x)dx\mathbb{P}[Z < \sqrt{\frac{(1-\rho)}{1+\rho}} x] \\
\mathbb{P}[\max(X,Y) \in dx] &= 2\phi(x)\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}} x\right) dx
\end{aligned}$$

where $\phi(x) := \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right)$, and $\Phi(x) := \int_{-\infty}^{x} \phi(t)dt$. Now let us compute its expectation:

$$\begin{aligned}
\mathbb{E}[\max(X,Y)] &= \int_{-\infty}^{\infty} 2x\phi(x)\Phi\left(\sqrt{\frac{1-\rho}{1+\rho}} x\right) dx \\
&= -2\int_{-\infty}^{\infty} u'(x)v(x)dx = [u(x)v(x)]_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} u(x)v'(x)dx
\end{aligned}$$

for $u(x) = \phi(x)$, $v(x) = \Phi\left(\sqrt{\frac{1-\rho}{1+\rho}} x\right)$, by integration by parts. Hence:

$$\begin{aligned}
\mathbb{E}[\max(X,Y)] &= \int_{-\infty}^{\infty} 2\phi(x)\sqrt{\frac{1-\rho}{1+\rho}}\phi\left(\sqrt{\frac{1-\rho}{1+\rho}} x\right) dx \\
&= 2\sqrt{\frac{1-\rho}{1+\rho}} \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp\{-\frac{x^2}{2}(1 + \frac{1-\rho}{1+\rho})\}dx \\
&= \sqrt{\frac{1-\rho}{1+\rho}} \frac{1}{\pi} \int_{-\infty}^{\infty} \exp\{-\frac{x^2}{2} \times \frac{2}{1+\rho}\}dx
\end{aligned}$$

Now, let $y = x\sqrt{\frac{2}{1+\rho}}$, by change of variables inside the integral we get finally:

$$\begin{aligned}
\mathbb{E}[\max(X,Y)] &= \sqrt{\frac{1-\rho}{1+\rho}}\frac{1}{\pi}\frac{\sqrt{1+\rho}}{\sqrt{2}}\int_{-\infty}^{\infty}\exp\{-\frac{y^2}{2}\}dy \\
&= \sqrt{\frac{1-\rho}{1+\rho}}\frac{1}{\pi}\frac{\sqrt{1+\rho}}{\sqrt{2}}\sqrt{2\pi} \\
\mathbb{E}[\max(X,Y)] &= \sqrt{\frac{1-\rho}{\pi}}
\end{aligned}$$