Problem 1.

(a). Since $f(x)$ is a density $1 = \int_{\mathbb{R}} f(x)\,dx$

$$\Rightarrow 1 = \int_{\mathbb{R}} f(x)\,dx$$

$$= \int_1^\infty \frac{c}{x^4}\,dx$$

$$= -\frac{c}{3x^3}\Big|_1^\infty = \frac{c}{3}$$

$\therefore c = 3$

(b) $$\mathbb{E}X = \int_{\mathbb{R}} x f(x)\,dx$$

$$= \int_1^\infty x \cdot \frac{3}{x^4}\,dx$$

$$= -\frac{3}{2}\frac{1}{x^2}\Big|_1^\infty = \frac{3}{2}$$

(c) $$\mathbb{E}X^2 = \int_{\mathbb{R}} x^2 f(x)\,dx$$

$$= \int_1^\infty x^2 \cdot \frac{3}{x^4}\,dx = -\frac{3}{x}\Big|_1^\infty = 3$$

$$\therefore \mathrm{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$= 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$$

Problem 2.

Note that $f(x) = \begin{cases} \frac{1}{2(1+x)^2} & x \geq 0 \\ \frac{1}{2(1-x)^2} & x < 0 \end{cases}$

(a) $$\mathbb{P}(-1 < X < 2) = \int_{-1}^2 f(x)\,dx$$

$$= \int_{-1}^0 \frac{1}{2(1-x)^2}\,dx + \int_0^2 \frac{1}{2(1+x)^2}\,dx$$

$$= \frac{1}{2(1-x)}\Big|_{-1}^0 + \left(-\frac{1}{2(1+x)}\right)\Big|_0^2 = \frac{7}{12}$$

(b) $$\mathbb{P}(|X| > 1) = \int_{-\infty}^{-1} f(x)\,dx + \int_1^\infty f(x)\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{1-x}\Big|_{-\infty}^{-1} + \frac{1}{2}\frac{-1}{(1+x)}\Big|_1^\infty = \frac{1}{2}$$

(c) $$\mathbb{E}|X| = \int_{-\infty}^\infty \frac{|x|\,dx}{2(1+|x|)^2} = \int_0^\infty \frac{x}{(1+x)^2}\,dx = \left[\ln(1+x) + \frac{1}{1+x}\right]_0^\infty = \infty$$

$\therefore \mathbb{E}X$ is not defined

Problem 3.

(a) $1 = \int_{\mathbb{R}} f(x)\,dx = \int_0^1 cx(1-x)\,dx$

$= c\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \frac{c}{6}$

$\therefore c = 6$

(b) X is continuous R.V

$\therefore \mathbb{P}(X = \frac{1}{2}) = 0$

(c) $\mathbb{P}(X < \frac{1}{2}) = \int_{-\infty}^{1/2} f(x)\,dx$

$= \int_0^{1/2} \frac{1}{6}x(1-x)\,dx$

$= 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^{\frac{1}{2}}$

$= \frac{1}{2}$

(d) $\mathbb{E}X = \int_0^1 6x^2(1-x)\,dx$

$= 2x^3 - \frac{3}{2}x^4 \Big|_0^1 = \frac{1}{2}$

$\mathbb{E}X^2 = \int_0^1 6x^3(1-x)\,dx$

$= \frac{3}{2}x^4 - \frac{6}{5}x^5 \Big|_0^1 = \frac{3}{10}$

$\therefore \mathrm{Var}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{20}$