Problem 1. Calls are poisson process with rate 1 per time

(a) $\underset{\sim}{\mathbb{P}(W_4} \leq 2) = 1 - \mathbb{P}(W_4 > 2)$

$= 1 - \exp(-2)$

time between 3rd and 4th call

$W_4 \sim \exp(1)$ $\therefore \mathbb{P}(W_4 > a) = e^{-a}$

(b) $\mathbb{P}(\underset{\sim}{T_4} \leq 5) = \mathbb{P}(\underset{\sim}{N_5} > 3) = 1 - e^{-5}\left(1 + 5 + \frac{25}{2} + \frac{125}{6}\right)$

time until 4th call

number of calls until t=5

$N_5 \sim \text{Pois}(5)$

(c) $\mathbb{E}(T_4) = \mathbb{E}(W_1 + W_2 + W_3 + W_4)$, $W_i \overset{iid}{\sim} \exp(1)$

$= 4\mathbb{E}(W_1)$

$= 4$

Problem 2.

$\mathbb{P}(Z = k) = \mathbb{P}(\lfloor T \rfloor = k)$

$= \mathbb{P}(k \leq T < k+1)$

$= \mathbb{P}(k \leq T) - \mathbb{P}(k+1 \leq T)$

$= e^{-\lambda k} - e^{-\lambda(k+1)}$

$= e^{-\lambda k}(1 - e^{-\lambda}) = (1-p)^k p$ where $p = 1 - e^{-\lambda}$

$\therefore Z \sim \text{geom}(1 - e^{-\lambda})$

Problem 7.

(a) $\Gamma(r+1) = \int_0^\infty x^r e^{-x} dx$

$= \int_0^\infty x^r (-e^{-x})' dx$

$= \left[-x^r e^{-x}\right]_0^\infty + \int_0^\infty r x^{r-1} e^{-x} dx$

$= 0 + r\Gamma(r)$

(b) $\Gamma(1) = \int_0^\infty e^{-x} dx = \left[-e^{-x}\right]_0^\infty = 1$

$\therefore \Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1) = (n-1)!$

(c) $T \sim Exp(1) \Rightarrow \mathbb{E}T^n = \int_0^\infty t^n e^{-t} dt$

$= \Gamma(n+1)$

$= n!$

(d) for any $t \geq 0$

$\mathbb{P}(S > t) = \mathbb{P}(T/\lambda > t)$

$= \mathbb{P}(T > \lambda t) = \int_{\lambda t}^\infty e^{-x} dx = e^{-\lambda t}$

$\therefore S \sim Exp(\lambda)$