Problem 1.

$$M_X(t) = \mathbb{E}(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}\left[x^2 - 2\mu x + \mu^2 - 2\sigma^2 t x\right]\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}\left[x^2 - 2(\mu+\sigma^2 t)x + \underbrace{\mu^2 + (\sigma^4 t^2 + 2\mu\sigma^2 t)}_{=(\mu+\sigma^2 t)^2} - (\sigma^4 t^2 + 2\mu\sigma^2 t)\right]\right)$$

$$= \exp(\mu t + \sigma^2 t^2/2) \underbrace{\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}\right)}_{\text{pdf of } N(\mu+\sigma^2 t,\ \sigma^2)} dx}_{=1}$$

$$= \exp(\mu t + \sigma^2 t^2/2) \qquad \text{for all } t \in \mathbb{R}$$

Problem 2.

(a) Let $X = Z_1 + \cdots + Z_n$ where $Z_i \overset{iid}{\sim} \mathrm{Bern}(p)$

$$\mathbb{E}e^{Z_i t} = e^0\, \mathbb{P}(Z_i = 0) + e^t\, \mathbb{P}(Z_i = 1) = pe^t + (1-p) \quad \text{for all } t \in \mathbb{R}$$

$$\therefore \mathbb{E}e^{Xt} = \mathbb{E}e^{(Z_1 + \cdots + Z_n)t}$$

$$= \mathbb{E}(e^{Z_1 t} \cdots e^{Z_n t})$$

$$= \mathbb{E}e^{Z_1 t}\, \mathbb{E}e^{Z_2 t} \cdots \mathbb{E}e^{Z_n t} \quad \} \text{ by independence}$$

$$= (1-p+pe^t)^n$$

(b) $$\frac{d}{dt}(1-p+pe^t)^n = pe^t \cdot n(1-p+pe^t)^{n-1}$$

$$\therefore \mathbb{E}X = \frac{d}{dt} M_X(t)\Big|_{t=0} = pe^0 \cdot n(1-p+pe^0)^{n-1} = np$$

Problem 3.

- $K(0) = \log M_X(0) = \log \mathbb{E}e^{0 \cdot X} = 0$

- $K'(t) = \frac{d}{dt} \log M_X(t) = \frac{M_X'(t)}{M_X(t)}$

$\therefore K'(t)|_{t=0} = \frac{M_X'(0)}{M_X(0)} = \mathbb{E}X$

- $K''(t) = \frac{d^2}{dt^2} \log M_X(t)$

$= \frac{d}{dt} \frac{M_X'(t)}{M_X(t)} = \frac{M_X''(t) M_X(t) - (M_X'(t))^2}{(M_X(t))^2}$

$\therefore K''(t)|_{t=0} = \frac{M_X''(0) M_X(0) - (M_X'(0))^2}{(M_X(0))^2} = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathrm{Var}\,X$