Problem 1.

let $Y = X^2$ $\quad \mathbb{P}(Y \le a)$ is $0$ if $a \le 0$. $\quad 1$ if $a \ge 4$.

for $0 < a \le 1$. $\quad \mathbb{P}(Y \le a) = \mathbb{P}(-\sqrt{a} \le X \le \sqrt{a}) = \frac{2\sqrt{a}}{3}$ ( $\because [-\sqrt{a}, \sqrt{a}] \subseteq [-1, 2]$ )

for $1 \le a \le 4$ $\quad \mathbb{P}(Y \le a) = \mathbb{P}(-\sqrt{a} \le X \le \sqrt{a})$

$= \mathbb{P}(-1 \le X \le \sqrt{a}) = \frac{\sqrt{a}+1}{3}$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{3\sqrt{y}} & y \in [0, 1] \\ \frac{1}{6\sqrt{y}} & y \in (1, 4] \\ 0 & \text{otherwise} \end{cases}$$

Problem 2.

(a) $Y > a \iff X_1, \cdots, X_n > a$

$\therefore \mathbb{P}(Y \le a) = 1 - \mathbb{P}(Y > a)$

$= 1 - \mathbb{P}(X_1 > a \,\&\, \cdots \,\&\, X_n > a)$

↓ by independence

$= 1 - \prod_{i=1}^{n} \mathbb{P}(X_i > a)$

$= 1 - e^{-\lambda n a}$

(b) $\therefore Y \sim \mathrm{Exp}(n\lambda)$ $\quad f_Y(y) = \lambda n e^{-\lambda n y}$

Problem 3.

(a) Range of $Y = [0, a/2]$

$$\mathbb{P}(Y \le y) = \begin{cases} 1 & \text{if } y \ge a/2 \\ 0 & \text{if } y \le 0 \\ y/a & \text{if } 0 \le y < \frac{a}{2} \end{cases}$$

(b) $\mathbb{E}Y = \mathbb{E}(\min(X, \frac{a}{2}))$

$= \int_0^a \frac{1}{a} \min(x, \frac{a}{2})\,dx = \int_0^{\frac{a}{2}} \frac{x}{a}\,dx + \int_{\frac{a}{2}}^{a} \frac{1}{a}\frac{a}{2}\,dx = \frac{a}{8} + \frac{a}{4} = \frac{3}{8}a$