Chap 2 Repeated trials and Sampling.

Counting

How many orderings of a, b, c, d, e are there?

$\underline{5}\,\underline{4}\,\underline{3}\,\underline{2}\,\underline{1} = 5!$

How many ordering of aabbb are there?

ex aabbb
ababb.

$$\binom{5}{2} = \frac{5!}{2!\,\cancel{3!}} = \frac{5\cdot 4}{2\cdot 1} = 10$$

$5! = 5\cdot 4\cdot \cancel{3\cdot 2\cdot 1}$

5 choose 2

n choose k ; $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ — # of orderings of k "a" and n-k "b".

$0 \le k \le n$

$\binom{n}{k} = \binom{n}{n-k}$

$0! = 1$

Find $\binom{20}{18} = \frac{20!}{18!\,2!} = \frac{20\cdot 19}{2\cdot 1}$

$\binom{20}{18} = \binom{20}{2}$

ex $(a+b)^2 = 1a^2 + 2ab + 1b^2$ — a sum of degree 2 monomials.

ex $(a+b)^5$ consists of sum of degree 5 monomials.
What is the coefficient of the a^3b^2 term?

Pascal's triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$a^2+2ab+b^2 \rightarrow$ 1 2 1

a^5 a^4b a^3b^2

$\binom{5}{3}$ — # of orderings of 3 "a" and 2 "b" when multiply out $(a+b)^5$.

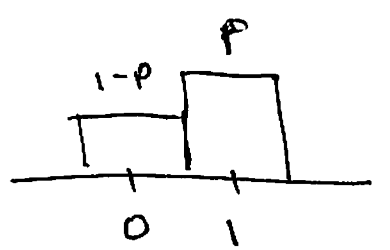
$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}$$

binomial expansion.

Bernoulli (p) distribution — success/failure trial

For $0<p\leq 1$ this is the distribution on $\{0,1\}$ defined by

outcome	0	1 ← success
prob	$1-p$ $= q$	p



ex Flip a trick coin (prob p heads)

head $=1$

tails $=0$.

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$a^5 \quad a^4 b \quad a^3 b^2$

$\binom{5}{3}$ — # of orderings of 3 a's and 2 b's when multiplying out $(a+b)^5$

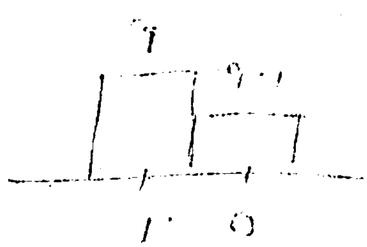
$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}}$$

binomial expansion.

Bernoulli (p) distribution — success/failure trial

For $0<p<1$ this is the distribution on $\{0,1\}$ determined by

outcome	0	1 (i.e. success)
prob	$1-p = q$	p



Ex: flip a fair coin (prob p heads)

heads = 1

tails = 0.

Binomial distribution

Suppose we have $\underline{n}$ indep Bernoulli (p) trials.

Possible number of successes? $0, 1, 2, \ldots, n$

ex What is chance of getting 2 sixes in n rolls?

S = roll a six — $1/6$
N = roll a non six — $5/6$

$\binom{5}{2} = 10$ orderings of 2 S and 3 N

SSNNN — prob $\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6}$.
NSSNN — prob $\frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \frac{5}{6}$

same.

$\vdots$

$$\Rightarrow P(2 \text{ sixes in } 5 \text{ rolls}) = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

$$\boxed{P(K \text{ successes in } n \text{ trials}) = \binom{n}{K} p^K (1-p)^{n-K}}$$

Binomial Formula.

ingredients

- n independent Bernoulli (p) trials.
- K is # successes.

ex Random # generator
Pick at random 20 numbers w/ replacement from $0, 1, 2, \ldots, 9$

$P(2$ 0's in 20 rows$)$ — $\binom{20}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18}$

$P(\text{more than } 2$ 0's$)$ — $1 - \sum_{K=0}^{2} \binom{20}{K}\left(\frac{1}{10}\right)^K\left(\frac{9}{10}\right)^{20-K}$

Binomial distribution

Suppose we have n indep Bernoulli(p) trials

Possible number of successes? $0, 1, 2, \ldots, n$

What is chance of getting 2 sixes in 5 rolls?

S = roll a six — $1/6$

N = roll a non-six — $5/6$

SSNNN — prob $\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}$

NSSNN — prob $\frac{5}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}$ same

$\binom{5}{2}$ arrangements of 2 S and 3 N

$P(\text{2 sixes in 5 rolls}) = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$

$$\boxed{P(k \text{ successes in } n \text{ trials}) = \binom{n}{k} p^k (1-p)^{n-k}}$$

Binomial Formula

Ingredients

- n independent Bernoulli(p) trials
- k is # successes

Ex: Random # generator

Pick at random 20 numbers w/ replacement from $0, 1, 2, \ldots, 9$

$P(\text{2 0s in 20 tries}) = \binom{20}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18}$

$P(\text{more than 2 0s}) = 1 - \sum_{k=0}^{2}\binom{20}{k}\left(\frac{1}{10}\right)^k\left(\frac{9}{10}\right)^{20-k}$

Consecutive odds ratio

$$R(k) = \frac{P(k)}{P(k-1)} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} = \boxed{\left(\frac{n-k+1}{k}\right)\frac{p}{q}}$$

$k \geq 1$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$= \left(\frac{n+1}{k} - 1\right) \cdot \frac{p}{q}$$

decreasing function.

If $R(k) = 1 \Rightarrow P(k) = P(k-1)$

≥ 1 $\qquad >$

< 1 $\qquad <$

ex $\text{Bin}(4, \tfrac{1}{2})$ $\quad P(k) = R(k) \cdot P(k-1)$

$n = 4$, $p = \tfrac{1}{2}$

$$P(0) = \binom{4}{0}\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^4 = \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}$$

$\binom{4}{0} = 1$

K	0	1	2	3	4
R(K)		$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$
P(K)	$\frac{1}{16}$	$\frac{1}{16}\cdot\frac{4}{1} = \frac{1}{4}$	$\frac{12}{32}$	$\frac{8}{32}$	$\frac{2}{32}$

Mode Use odds ratio to find most likely number of successes (mode).

i.e. looking for k with biggest $P(k)$.

$$P(k-1) \leq P(k)$$

$$\Leftrightarrow \quad 1 \leq P(k)/P(k-1) = R(k)$$

$$\Leftrightarrow \quad 1 \leq \frac{n-k+1}{k} \cdot \frac{p}{1-p}$$

(Look at the ratio of successive probabilities)

$$R(k) = \frac{b(k)}{b(k-1)} \underset{k \ge 1}{=} \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}}$$

$\frac{n!}{k!(n-k)!}$

$$= \left(\frac{n+1}{k} - 1 \right) \frac{p}{q}$$

decreasing in k

If $R(k) = 1 \Rightarrow b(k) = b(k-1)$

$\ge 1 \qquad >$

$< 1 \qquad <$

Ex: $\mathrm{Bin}(4, \tfrac{1}{2})$ $\quad b(k) = R(k) \cdot b(k-1)$

$b(0) = (\tfrac{1}{2})^4 = \frac{1}{16}$

k	0	1	2	3	4
$R(k)$		$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$
$b(k)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$\frac{1}{4}$

<u>Note</u>: use ratio to find most likely number of successes (mode).

We're looking for k with biggest $b(k)$.

$b(k-1) \le b(k)$

$\Leftrightarrow \quad 1 \le b(k)/b(k-1) = R(k)$

$\Leftrightarrow \quad 1 \le \frac{n-k+1}{k} \cdot \frac{p}{1-p}$

$\Leftrightarrow \quad k(1-p) \le (n-k+1)p$

$\Leftrightarrow \quad \boxed{k \le np + p = (n+1)p}$

we will continue this next time.