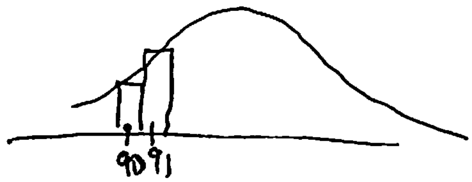
Stat 134 lec 6

Normal approx to binomial

Find P(more than 90 sixes in 600 rolls of a die)



normal approx

$\mu = np = 600\left(\frac{1}{6}\right) = 100$

$\sigma = \sqrt{npq} = \sqrt{600\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)} = 9.13$

$1 - \Phi\left(\frac{90.5 - 100}{9.13}\right) = .85$

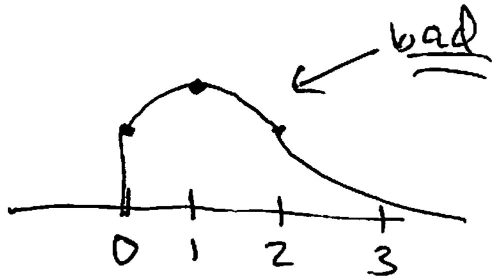
When not to use normal approx to binomial

Bin(1000, .001)

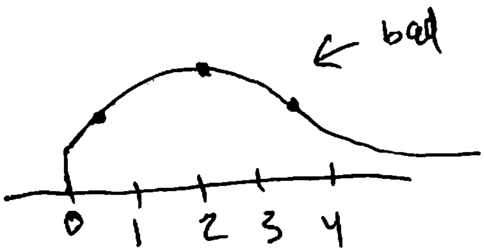
Find P(more than 1 success)

$\mu = np = 1000\left(\frac{1}{1000}\right) = 1$

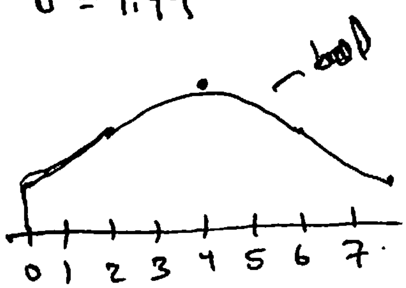
$\sigma = \sqrt{npq} \approx 1$



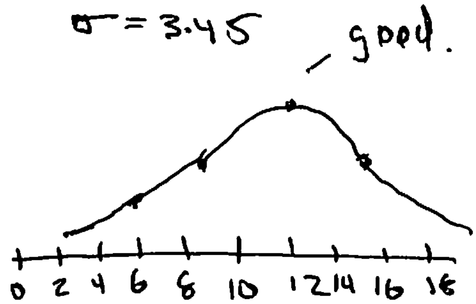
$n = 200$
$p = .01$
$\mu = 2$
$\sigma = \sqrt{2(.99)} = 1.41$



$n = 400$
$p = .01$
$\mu = 4$
$\sigma = 1.99$



$n = 1200$
$p = .01$
$\mu = 12$
$\sigma = 3.45$



For fixed p, ↑n the normal approx looks better.

Note: More rigorously, If n is large and $\underline{\sigma > 3}$ use normal approx.

we need $\mu > 3\sigma \Leftrightarrow (np)^2 > 9npq \Leftrightarrow \sigma > 3q$

but $\mu > 3 \Rightarrow \mu > 3q$ since $0 < q < 1$.

Law of large #s

Proportion of successes $= \dfrac{\#\text{ successes}}{n}$

between 0 and 1

$$\text{avg} = \frac{\text{avg}(\#\text{ successes})}{n} = \frac{np}{n} = p$$

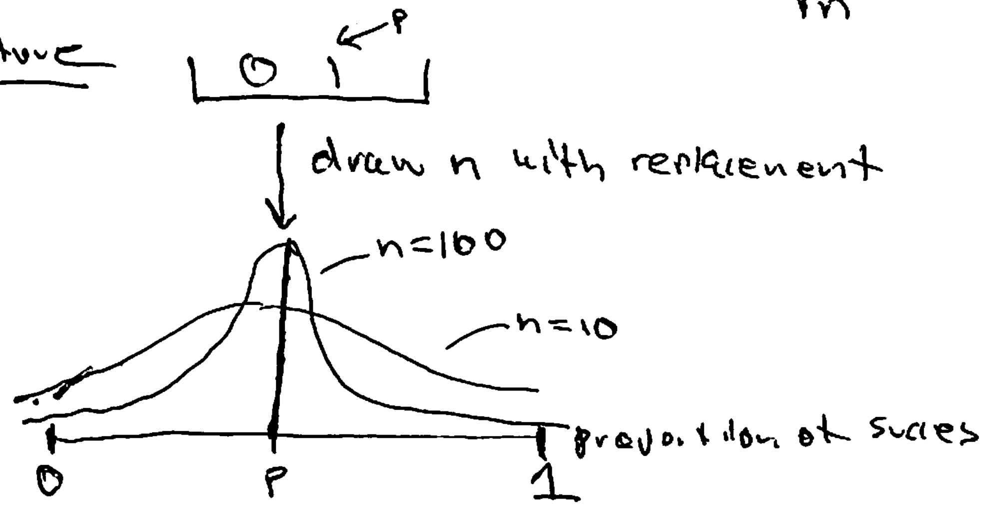
$$\sigma = \frac{\sigma(\#\text{ successes})}{n} = \frac{\sqrt{npq}}{n} = \frac{\sqrt{pq}}{\sqrt{n}}$$

square root law for proportion of success,

says,

Proportion of success with high probability lie in a small interval centered on p, with width a moderate multiple of $\frac{1}{\sqrt{n}}$

Picture



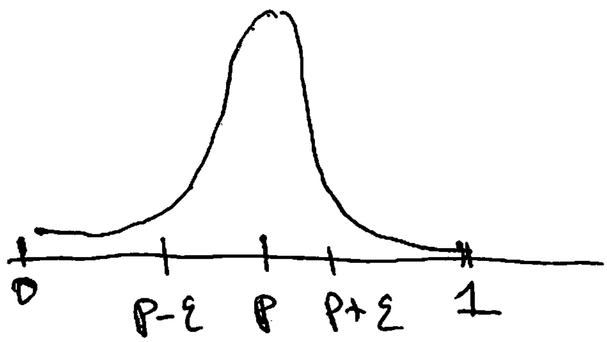
Law of large #s,

In a large number of independent Bernoulli (p) trials (i.e. n is large) the proportion of success is likely to be p.

More formally

Fix $\varepsilon > 0$

$P(\text{prop of successes is in range } p \pm \varepsilon) \to 1$ as $n \to \infty$.



I-clicker question (next page)

1. A fair coin is tossed, and you win a dollar if there are exactly 50% heads. Which is better?

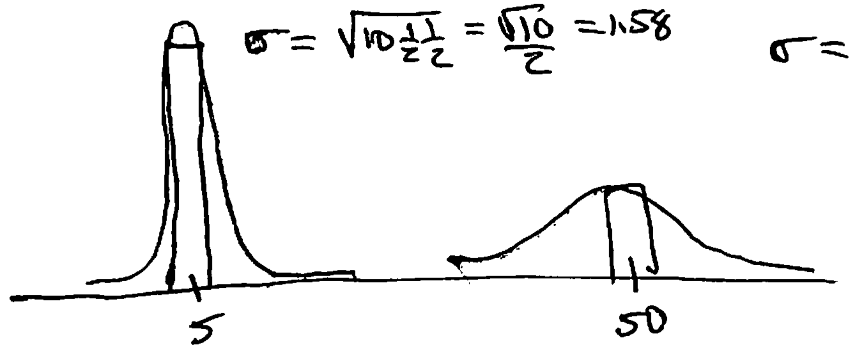
a 10 tosses

b 100 tosses

exactly $\binom{10}{5}\left(\frac{1}{2}\right)^{10} = .246$ $\qquad \binom{100}{50}\left(\frac{1}{2}\right)^{100} = .08$

Correct picture

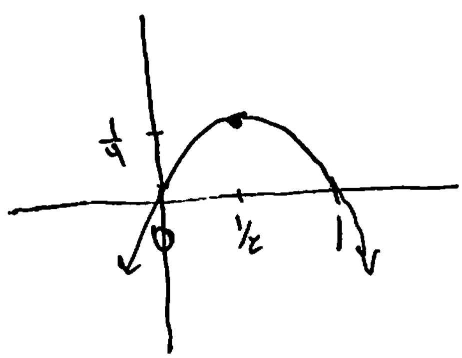
$\sigma = \sqrt{10 \frac{1}{2}\frac{1}{2}} = \frac{\sqrt{10}}{2} = 1.58$ $\qquad \sigma = \sqrt{100 \frac{1}{2}\frac{1}{2}} = \frac{10}{2} = 5$



Poisson approximation (sec 2.4 (skip 2.3))

let $f(p) = p(1-p) = -p^2 + p$

Graph of $f(p)$



So $\sigma = \sqrt{npq}$ is small for p near 0 or 1.

So don't use normal approx for p near 0, 1.

we will use Poisson approx when n large and p small.

derivation of Poisson distribution

The Poisson(μ) distribution is a limit of Binomials.

Consider $Bin(n, p_n)$

probability changes with n.

where $p_n \to 0$ and $np_n \to \mu$ as $n \to \infty$

some positive #

see example (next page)

$P_n(k)$ = prob of k successes at stage n

$$= \binom{n}{k} p_n^k (1-p_n)^{n-k}$$

Prob of success.

$$P_n(0) = (1-p_n)^n.$$

p_n small

$$\log_e P_n(0) = n \log_e(1-p_n) \approx n(-p_n)$$

$$\Rightarrow P_n(0) = e^{-np_n} \approx e^{-\mu}$$

use odds ratios to find $P_n(k)$

$$P_n(k) = P_n(k-1) R(k) = P_n(k-1) \frac{n-k+1}{k} \cdot \frac{p_n}{1-p_n}$$

$$= P_n(k-1) \left(\frac{\overset{\to \mu}{np_n}}{k} - \underbrace{\frac{(k-1)p_n}{k}}_{\to 0} \right) \underbrace{\frac{1}{1-p_n}}_{\to 1}$$

$$\approx \boxed{P_n(k-1) \frac{\mu}{k}}$$

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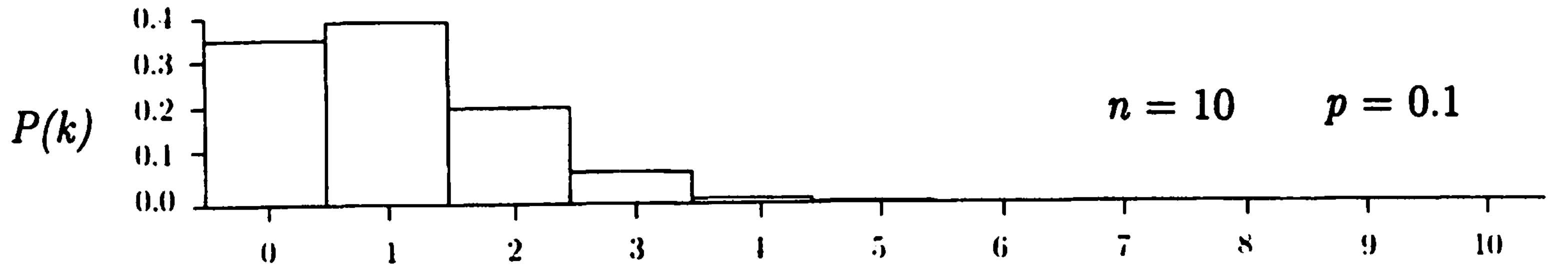
Example 1. The binomial (10, 1/10) distribution.

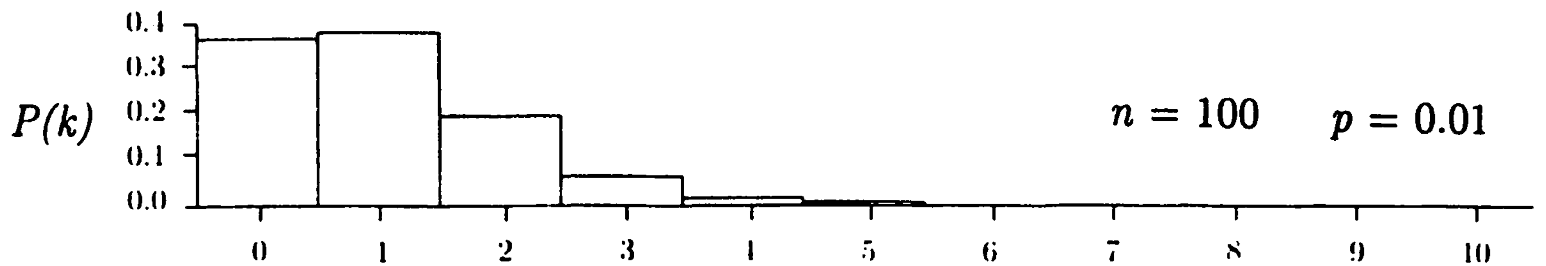
Poisson($\mu=1$) is a limit of binomials $Bin(n, \frac{1}{n})$

This is the distribution of the number of black balls obtained in 10 random draws with replacement from a box containing 1 black ball and 9 white ones.



Example 2. The binomial (100, 1/100) distribution.

This is the distribution of the number of black balls obtained in 100 random draws with replacement from a box containing 1 black ball and 99 white ones.



Example 3. The binomial (1000, 1/1000) distribution.

Now take 1000 random draws with replacement from a box with 1 black ball and 999 white ones. This is the distribution of the number of black balls drawn: