Poisson Distribution

The Poisson distribution Poisson (μ) is a limit of a sequence of binomials $Bin(n, p_n)$ with

$p_n \to 0$ and $np_n \to \mu$ as $n \to \infty$

we found that at n^{th} stage of this sequence of binomials

$P_n(0) \approx e^{-\mu}$ see example next page.

and $P_n(k) = P_n(k-1)\frac{\mu}{k}$

$P_n(1) = e^{-\mu} \cdot \frac{\mu}{1}$

$P_n(2) = e^{-\mu}\frac{\mu}{1} \cdot \frac{\mu}{2} = e^{-\mu}\frac{\mu^2}{2\cdot 1}$

$P_n(3) = e^{-\mu}\frac{\mu^3}{3!}$

$\vdots$

$$\boxed{P_n(k) = e^{-\mu}\frac{\mu^k}{k!} \quad \text{where } \mu \approx np_n}$$

Def^n Poisson (μ) distribution

$P_\mu(k) = e^{-\mu}\frac{\mu^k}{k!}$

Does this add up to 1? $\sum_{k=0}^{\infty}\frac{\mu^k}{k!} = e^{\mu}$

So $\sum_{k=0}^{\infty} P_\mu(k) = \sum_{k=0}^{\infty} e^{-\mu}\frac{\mu^k}{k!} = e^{-\mu}\underbrace{\left(\sum_{k=0}^{\infty}\frac{\mu^k}{k!}\right)}_{=e^{\mu}} = 1$

So $Bin\left(10{,}000, \frac{2}{10{,}000}\right)$ — use Poisson (2)

What if # successes is Binomial $\left(100, \frac{99}{100}\right)$?

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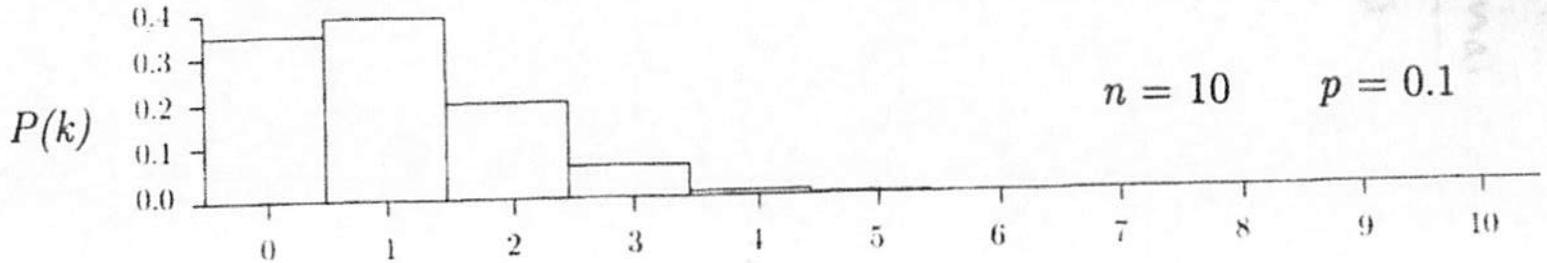
Example 1. **The binomial (10, 1/10) distribution.**

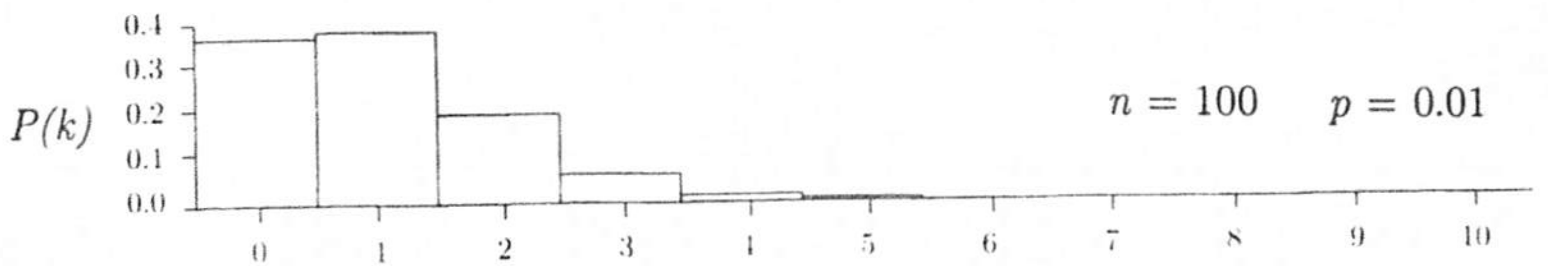
This is the distribution of the number of black balls obtained in 10 random draws with replacement from a box containing 1 black ball and 9 white ones.

Poisson ($\mu=1$) is a limit of binomials $Bin(n, \frac{1}{n})$



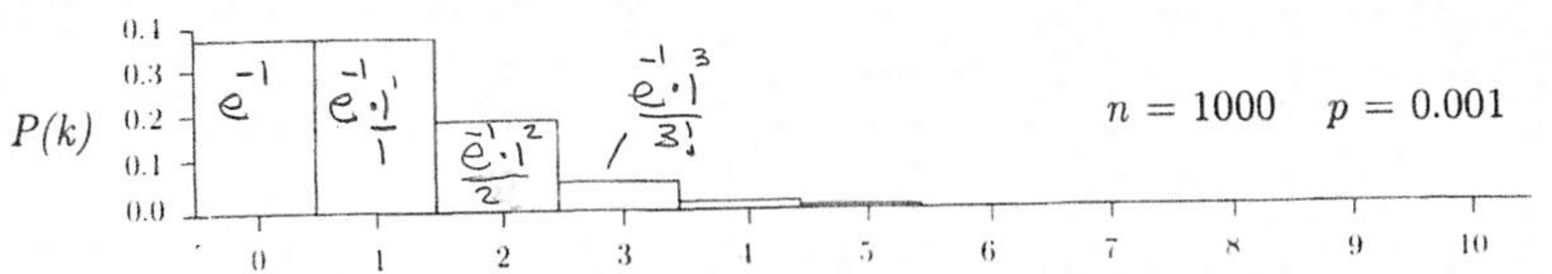
Example 2. **The binomial (100, 1/100) distribution.**

This is the distribution of the number of black balls obtained in 100 random draws with replacement from a box containing 1 black ball and 99 white ones.



Example 3. **The binomial (1000, 1/1000) distribution.**

Now take 1000 random draws with replacement from a box with 1 black ball and 999 white ones. This is the distribution of the number of black balls drawn:



The # failures is Binomial $\left(100, \frac{1}{100}\right)$ ← approx by Poisson (1)

$$P(\leq 40 \text{ successes}) = P(\geq 60 \text{ failures}) = \sum_{k=60}^{100} e^{-1} \frac{1^k}{k!}.$$

ex Bet 500 times, independently, on a bet with $\frac{1}{1000}$ chance of winning. $\mu = \frac{1}{2}$

large

small

What is chance of winning more than 1 bet?

$$P_\mu(k) = \frac{e^{-\mu}\mu^k}{k!} = \frac{e^{-\frac{1}{2}} \frac{1}{2}^k}{k!}$$

$$P(\text{more than 1 win}) = 1 - P(0 \text{ or } 1 \text{ win})$$

$$= 1 - \left(e^{-\frac{1}{2}} + e^{-\frac{1}{2}} \frac{1}{2}\right) = .0902$$

ex

97.8% of approx 30 million poor families in the US have a fridge. If you randomly sample 100 of these families, roughly what is the chance that 98 or more have a fridge?

$$P(98 \text{ or more fridge}) = P(\leq 2 \underbrace{\text{don't have a fridge}}_{.022})$$

let $p = .022$

$n = 100$

$$= e^{-2.2} + e^{-2.2}(2.2) + \frac{e^{-2.2}(2.2)^2}{2} = .6227 \leftarrow \text{easy to calculate!}$$

Note: # people with a fridge is also binomial (100, .978)

$$P(98 \text{ or more fridge}) = \binom{100}{98}(.978)^{98}(.022)^2 + \binom{100}{99}(.978)^{99}(.022)^1 + \binom{100}{100}(.978)^{100}$$

painful to calculate! $\rightarrow = .6221$

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Random Sampling

Given a composition of a population we will ask questions about the composition of a sample.

Sampling with replacement

ex Class A: 50%
B: 30%
C: 15%
D: 5%

sample 20 students at random with replacement

Find P(8 A's, 6 B's, 4 C's, 2 D's) ?

$$= \binom{20}{8}\cdot\binom{12}{6}\cdot\binom{6}{4}(.5)^8(.3)^6(.15)^4(.05)^2$$

$$= \frac{20!}{8!\,\cancel{12!}}\cdot\frac{\cancel{12!}}{6!\,\cancel{6!}}\cdot\frac{\cancel{6!}}{4!\,2}(.5)^8(.3)^6(.15)^4(.05)^2$$

$$= \frac{20!}{8!\,6!\,4!\,2!}(.5)^8(.3)^6(.15)^4(.05)^2$$

Multinomial formula

$$\binom{20}{8,6,4,2}$$

Multinomial distribution

n trials, events are divided into 1, 2, .., k categories with prob $p_1, p_2, \dots, p_k$

Draw x_k for each category.

$$\boxed{P = \frac{n!}{x_1!\,x_2!\cdots x_k!}\,p_1^{x_1}\cdots p_k^{x_k}}$$

generalization of binomial $k=2$.

$$\binom{7}{3,4} = \frac{7!}{3!\,4!}$$

$$\binom{7}{3} = \binom{7}{4}$$

Sampling without replacement

ex 52 card standard deck

A 5 card poker hand consists of 5 cards dealt at random without replacement (Simple random Sample SRS)

ex

P (a particular poker hand) ?

$$\# \text{ poker hands } \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5!} = \frac{52!}{5!\,47!} = \binom{52}{5}$$

$$= \boxed{\frac{1}{\binom{52}{5}}}$$ since each poker hand is equally likely,

ex Deck contains 4 aces.

P (poker hand has 2 aces) ?

$$\frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$$ ← hypergeom formula.