Stat 134 lec 11

2/9/18

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Sum of Two RV

ex $X \sim Pois(\mu)$ — $0, 1, 2, 3, \ldots$
$Y \sim Pois(\lambda)$ — $0, 1, 2, 3, \ldots$
independent,

$S = X + Y$ Find dist of S. — pos values $0, 1, 2, 3, \ldots$

For $s \geq 0$
Find $P(S=s)$.

Pick up next time.

Cont: addn rule

$$P(S=s) = P(X=0, Y=s) + P(X=1, Y=s-1) + \cdots P(X=k, Y=s-k) + \cdots + P(X=s, Y=0)$$

$$= \sum_{k=0}^{s} P(X=k, Y=s-k)$$

$$= \sum_{k=0}^{s} P(X=k)P(Y=s-k) \quad \text{independence of } X, Y.$$

$$= \sum_{k=0}^{s} \frac{e^{-\lambda}\lambda^{k}}{k!} \frac{e^{-\mu}\mu^{s-k}}{(k-s)!}$$

$\binom{s}{k}$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} \sum_{k=0}^{s} \frac{s!}{k!(k-s)!} \lambda^{k} \mu^{s-k}$$

Stupid trick, I multiply numerator and denominator by $s!$

$$= e^{-(\lambda+\mu)} \frac{1}{s!} (\lambda+\mu)^{s}$$

(recall binomial thm $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$)

$\Rightarrow S \sim Pois(\lambda+\mu)$.

The sum of two independent Poisson is Poisson!

1.5

2. Which is correct

A pair of dice are thrown. The total number of spots is like

(i) one draw from the box

2	3	4	5	6	7	8	9	10	11	12

(ii) the sum of two draws from the box

1	2	3	4	5	6

a (i)

b (ii)

$X_1 \sim \text{Bin}\left(1000, \frac{1}{1000}\right)$ — $\mu = 1$ — Pois(1)

indep.

$X_2 \sim \text{Bin}\left(2000, \frac{1}{1000}\right)$ — $\mu = 2$ — Pois(2)

$X_1 + X_2 \sim \text{Bin}\left(3000, \frac{1}{1000}\right)$ — Pois(3) — $\mu = 1 + 2 = 3$

Expectation

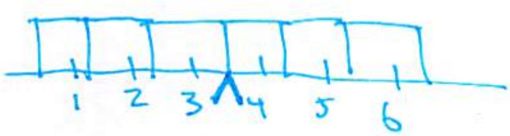
$$E(X) = \sum_{\omega \in \Omega} X(\omega) P(\omega) = \sum_{\text{all } x \text{ in } X} x \cdot P(X = x)$$

ex roll a die

X = face value — 1, 2, 3, 4, 5, 6 $\quad X \sim \text{Unif}\{1,2,3,4,5,6\}$

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \frac{1}{6} + 5 \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{1}{\cancel{6}}\left(\frac{7 \cdot \cancel{6}}{2}\right) = 3.5$$



More generally

$X \sim \text{Unif}\{1, 2, \dots, n\}$

$$E[X] = \frac{1}{\cancel{n}}\left(1 + \cdots + n\right) = \frac{n+1}{2}$$

$$1 + \cdots + n = \frac{(n+1)\cancel{n}}{2}$$

ex $X = 10$

$E(X) = 10 \cdot 1 = 10$

$$\boxed{E(c) = c}$$

Additivity

If X and Y are defined on the same space

then $E(X+Y) = E(X) + E(Y)$.

sum over all $\omega \in \Omega$

$$(X+Y)(\omega) = X(\omega) + Y(\omega)$$

$$(X+Y)(\omega) \cdot P(\omega) = X(\omega) \cdot P(\omega) + Y(\omega) \cdot P(\omega).$$

It doesn't matter if X, Y are dependent or indep.

Linear func rule

Let $W = aX$ $\quad E(W) = aE(X)$

$Y = aX + b$ $\quad E(Y) = \underbrace{E(aX)}_{= aE(X)} + \underbrace{E(b)}_{= b}$.

$$E(\underbrace{3 - 4X + Y}) = 3 - 4E(X) + E(Y)$$

linear for this to work.

Indicators

ex $X \sim \mathrm{Bin}(n, p)$

$X = I_1 + I_2 + \cdots + I_n$ where $I_j = \begin{cases} 1 & \text{if } j^{th} \text{ trial is success} \quad \leftarrow p \\ 0 & \text{else} \quad \leftarrow 1-p \end{cases}$

successes in n trials.

$E(I_j) = 1 \cdot p + \cancel{0 \cdot (1-p)} = p$ for all j

$$E(X) = E(I_1) + E(I_2) + \cdots + E(I_n) = \boxed{n \cdot p}$$

ex $X \sim \text{Hypergeom}(N, 6, n)$ — $\frac{6}{N}$

$X = I_1 + \cdots + I_n$ where $I_j = \begin{cases} 1 & \text{if draw } j \text{ is good} \\ 0 \end{cases}$

$$E(I_j) = \frac{6}{N}$$

$$\boxed{E(X) = n\frac{6}{N}}$$

$N = 52$
$6 = 4$
$n = 5$

Expected # aces in a poker hand?

$$E(\#\text{ aces}) = \boxed{5 \cdot \frac{4}{52}}$$

Notice indicators are dependent,

* For expec~~pect~~ed number of ... ⇒ use indicators

More generally,

Suppose X is the number of event that occur among some collection of events $A_1, \cdots, A_n$

then $E(X) = P(A_1) + \cdots + P(A_n)$

↑

$E(I_{A_1})$ where $I_{A_j} = \begin{cases} 1 & \text{if } A_j \\ 0 & \text{else.} \end{cases}$

ex Suppose a fair die is rolled 10 times, Find the expected number of different faces that appear in 10 rolls.

1,2,3,4,5,6 — X = number of diff faces that appear in 10 rolls,

A_1? — face 1 appears at least once ← $1 - \left(\frac{5}{6}\right)^{10}$

$I_{A_1} = \begin{cases} 1 & \text{if } A_1 \\ 0 & \text{else} \end{cases}$ — $1 - \left(\frac{5}{6}\right)^{10}$

$X = I_{A_1} + I_{A_2} + \cdots I_{A_6}$ ⇒ $\boxed{E(X) = 6\left(1 - \left(\frac{5}{6}\right)^{10}\right)}$

Expectation of a function of a RV

Last time we did example

X = # heads when flip fair coin twice

$g(x) = |x-1|$

x	0	1	2
$P(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
$g(x)$	1	0	1
W	0	1	1

$W = \min(x,1)$

$$E(g(x)) = E(|x-1|)$$

$$= 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = \frac{1}{2}$$

More generally

$$\boxed{E(g(x)) = \sum_{x \in X} g(x) \cdot P(x)}$$

ex $E(W) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 3/4$