Stat 134 Lecture (Fri 3/9/17): 4.2 Poisson Process

- [ ] Exponential Distribution (Review)

$T \sim Exp(\lambda)$
$f_T(t) = \lambda e^{-\lambda t}$
$P(T > t) = e^{-\lambda t}$
$F_T(t) = 1 - e^{-\lambda t}$
$E(T) = \frac{1}{\lambda}$
$Var(T) = \frac{1}{\lambda^2} \Rightarrow SD(T) = \frac{1}{\lambda}$

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1. Suppose $T \sim Exp(\lambda = 2\ s^{-1})$. Find:

a) $f_T(t)$

$f_T(t) = \lambda e^{-\lambda t} = 2e^{-2t}$

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b) $P(T > 7)$ and $P(T < 7)$

$P(T > t) = e^{-\lambda t} \Rightarrow P(T > 7) = e^{-2\cdot 7} = e^{-14}$
$P(T < t) = 1 - e^{-\lambda t} \Rightarrow P(T < 7) = 1 - e^{-14}$

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c) $P(7 < T < 12)$

$P(7 < T < 12) = P(T > 7) - P(T > 12) = e^{-2\cdot 7} - e^{-2\cdot 12} = e^{-14} - e^{-24}$

- [ ] Median of Exponential Distribution

$$t_H = \frac{1}{\lambda}\log 2$$

Pf:
$F_T(t) = 1 - e^{-\lambda t}$
$F_T(t_H) = 1 - e^{-\lambda t_H} = \frac{1}{2} \Rightarrow e^{-\lambda t_H} = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow -\lambda t_H = \log\left(\frac{1}{2}\right) \Rightarrow \lambda t_H = \log 2$
$t_H = \frac{1}{\lambda}\log 2$

- [ ] Memoryless Property

$P(T > w + t | T > w) = P(T > t) = e^{-\lambda t}$
Pf:
Suppose $T \sim Exp(\lambda)$. Prove T is memoryless.

$$P(T > w + t | T > w) = e^{-\lambda t}$$

$P(T > w + t | T > w) = \frac{P(T > w + t)}{P(T > w)} = \frac{\lambda e^{-\lambda(t+w)}}{e^{-\lambda w}} = e^{-\lambda t}$

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2. Suppose $T \sim Exp(\lambda)$. Let's try to interpret λ. Find:

a) $f_T(0)dt \Leftrightarrow P(0 < T < 0 + dt)$

$f_T(0)dt = \lambda e^{-\lambda(0)}dt = \lambda dt$

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b) $P(T \in dt) \Leftrightarrow P(t < T < t + dt) = f_T(t)dt$

$f_T(t)dt = \lambda e^{-\lambda t}dt$

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c) $P(T \in dt | T > t)$

$P(T \in dt | T > t) = \frac{P(T \in dt)}{P(T > t)} = \frac{\lambda e^{-\lambda t}dt}{e^{-\lambda t}} = \lambda dt$

This tells us that λ is the instantaneous rate that an event occurs such as an arrival (or death). This is another proof that the exponential is memoryless. In fact, it is the only density that is memoryless. Geometric is the only discrete distribution that is memoryless.

☐ Bernoulli Trials

On page 288-289, the text draws important parallels between the Poisson Arrival Process and Bernoulli trials.

$I_i \sim Bern(p)$

$W_i \sim iid\ Geom(p)$

$T_r \sim NegBin(r,p)$

$X \sim Bin(n,p)$

Example:

1	2	3	4	5	6	7	8	9	10
F	F	S	F	F	F	S	S	F	S
		$w_1 = 3$				$w_2 = 4$	$w_3 = 1$		$w_4 = 2$
		$t_1 = 3$				$t_2 = 7$	$t_3 = 8$		$t_4 = 10$
$X \sim Bin(n = 10, p): x = 4$									

☐ Poisson Arrival Process

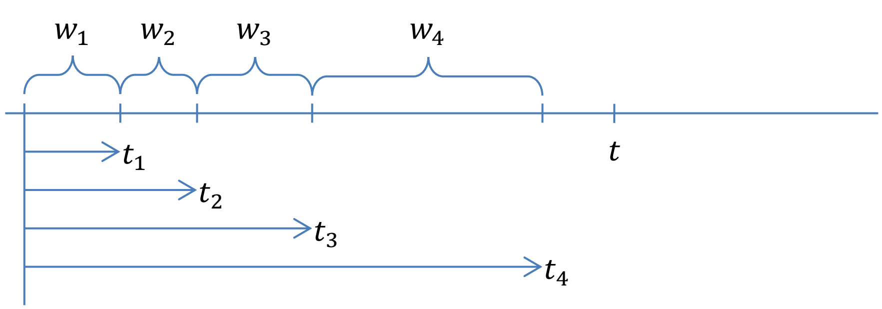
$$T \sim PP(\lambda) \Rightarrow \begin{cases} W_i \sim iid\ Exp(\lambda) \\ T_r \sim Gamma(r,\lambda) \\ N_{(0,t]} \sim Pois(\mu = \lambda t) \end{cases}$$

Notice that the W_i's are independent, but the T_r's are dependent.

Underlying assumptions:

- $N(0) = 0$.
- Disjoint time intervals are independent.
- Number of events depends only on the length (not location).

Example:



☐ Relating a Positive Integer r Gamma to Sum of Exponentials

$T = W_1 + \cdots + W_r$ where $W_i \sim iid\ Exp(\lambda)$

$T \sim Gamma(r,\lambda)$ ■

☐ $E(T)$ and $Var(T)$ of a gamma density

Don't need density to find $E(T)$ and $Var(T)$. Express T as a function of simpler RVs.

$$E(T) = E(W_1 + \cdots + W_r) = rE(W_1) = \frac{r}{\lambda}$$ ■

$$Var(T) = Var(W_1 + \cdots + W_r) = rVar(W_1) = \frac{r}{\lambda^2}$$ ■

☐ Limit of T as $r \to \infty$.

Draw diagram.

$$r \to \infty: T \sim N\left(\frac{r}{\lambda}, \frac{r}{\lambda^2}\right)$$ ■

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3. Give a density argument to derive the gamma density for an integer r.
Let $N \sim Pois(\lambda t)$ and $W \sim Exp(\lambda)$.

$$P(T \in dt) = P(N = r-1)P(W \in dt | W > t) = e^{-\lambda t}\frac{(\lambda t)^{r-1}}{(r-1)!} \cdot \lambda dt = \frac{1}{(r-1)!}\lambda^r t^{r-1} e^{-\lambda t} dt$$

If $r \in \mathbb{Z}^+$, then $\Gamma(r) = (r-1)!$.
Replace $(r-1)!$ with $\Gamma(r)$ to account for all positive real numbers.

$$f_T(t) = \underbrace{\frac{1}{\Gamma(r)}\lambda^r}_{\text{normalizing constant}} \underbrace{t^{r-1}e^{-\lambda t}}_{\text{functional form}} = \frac{1}{\Gamma(r)}\lambda^r t^{r-1} e^{-\lambda t} \blacksquare$$

Gamma Function
For non-integer values of r, show $\Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt$.

$f_T(t) = \frac{1}{\Gamma(r)}\lambda^r t^{r-1}e^{-\lambda t}$

Set $\lambda = 1$.

$f_T(t) = \frac{1}{\Gamma(r)}t^{r-1}e^{-t}$

$$1 = \int_0^\infty \frac{1}{\Gamma(r)}t^{r-1}e^{-t}dt \Rightarrow \Gamma(r) = \int_0^\infty t^{r-1}e^{-t}dt \blacksquare$$

4. Suppose $T \sim Gamma(r = 4, \lambda = 2\ s^{-1})$. Find:

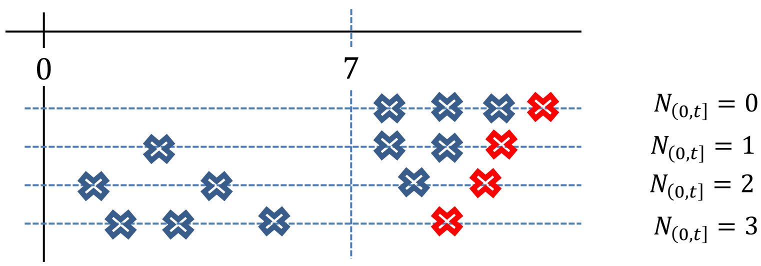
a) $f_T(t)$

$$f_T(t) = \frac{1}{\Gamma(r)}\lambda^r t^{r-1}e^{-\lambda t} = \frac{1}{\Gamma(4)}2^4 t^3 e^{-2t} = \frac{8}{3}t^3 e^{-2t}$$

b) $P(T > 7)$ and $P(T < 7)$

$$P(T > t) = \int_t^\infty \frac{1}{\Gamma(r)}\lambda^r s^{r-1}e^{-\lambda s}ds = P\left(N_{(0,t]} \le r-1\right) = \sum_{i=0}^{r-1} e^{-\lambda t}\frac{(\lambda t)^i}{i!}$$

In an interval from $(0, 7]$, we want at most $4 - 1$ arrivals.



$N_{(0,t]} \sim Pois(\mu = \lambda t = 2 \cdot 7 = 14)$

$$P(T > 7) = e^{-14}\left[\frac{14^0}{0!} + \frac{14^1}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!}\right] = \frac{1711}{3}e^{-14}$$

$$P(T < 7) = 1 - P(T > 7) = 1 - \frac{1711}{3}e^{-14}$$