warmup: 11:00–11:10 Stat 134 lec 11

A drawer contains s black socks and s white socks (s> 0). I pull two socks out at random without replacement and call that my first pair. Then I pull out two socks at random without replacement and call that my second pair. I proceed in this way until I have s pairs and the drawer is empty. Find the expected number of pairs in which two socks are different colors.

X = number of pairs (out of S) of mismatches.

$$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ pair is mismatch} \\ 0 & \text{else} \end{cases}$$

p

$P = 2\frac{s}{2s}\cdot\frac{s}{2s-1}$ ← change get WB or BW — or $\frac{\binom{s}{1}\cdot\binom{s}{1}}{\binom{2s}{2}}$

$X = I_1 + \dots + I_s$

$$\boxed{E(X) = s\cdot\frac{\binom{s}{1}\cdot\binom{s}{1}}{\binom{2s}{2}}}$$

Last time Sec 3.2 Expectation

$$E(X) = \sum_{x \in X} x P(X=x)$$

If X is a count, X can be written as a sum of indicators

$$X = I_1 + I_2 + \cdots + I_n, \quad I_j = \begin{cases} 1 & \text{prob } p \\ 0 & \text{prob } 1-p \end{cases}$$

$1 \le j \le n$

$$E(I_j) = 1 \cdot p + 0 \cdot (1-p) = p.$$

Idea Even if indicators are dependent the expectation of each indicator is an unconditional probability.

Try choosing indicators such that all indicators have the same expectation p.

Then $E(X) = n \cdot p$

We proved if $X \sim Bin(n,p) \Rightarrow E(X) = np$

if $X \sim HG(n, N, G) \Rightarrow E(X) = n\frac{G}{N}$

ex X = # aces in a poker hand from a deck of cards

$X \sim HG(n, N, G)$ $N = 52$, $G = 4$, $n = 5$

$X = I_1 + I_2 + I_3 + I_4 + I_5$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ card is an ace} \\ 0 & \text{else} \end{cases}$ — $p = 4/52$

$E(X) = 5 \cdot E(I_1) = \boxed{5 \cdot \left(\frac{4}{52}\right)}$

Today

1. Sec 3.2 More expectation with indicator examples
2. Sec 3.2 tail sum formula

① **Sec 3.2** **more expectation / indicator examples**

ex Consider a 5 card deck consisting of 2, 2, 3, 4, 5.
Shuffle the cards.
Let X = number of cards before the first 2.

a) What are the range of values of X?

$0, 1, 2, 3$

b) Write X as a sum of indicators

$X = I_3 + I_4 + I_5$

c) How is an indicator defined.

P

$$I_3 = \begin{cases} 1 & \text{if 3 before first 2} \\ 0 & \text{else} \end{cases}$$

d) Find $E(I_3)$

$\frac{1}{3}$

slot can be empty or have a 3 in it.

$\frac{1}{3}$

___ 2 ___ 2 ___

3 places you can put 3

e) Find $E(X)$

$$E(X) = \underbrace{E(I_3)}_{= \frac{1}{3}} + \underbrace{E(I_4)}_{= \frac{1}{3}} + \underbrace{E(I_5)}_{= \frac{1}{3}} = \boxed{1}$$

Stat 134

1. Consider a well shuffled deck of cards. The expected number of cards before the first ace is?

   a 52/5

   b 48/5

   c 48/4

   d none of the above

$X = I_1 + I_2 + \cdots + I_{48}$

p

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ nonace before } 1^{st} \text{ ace} \\ 0 & \text{else} \end{cases}$

$\_\_ \; A_1 \; \_\_ \; A_2 \; \_\_ \; A_3 \; \_\_ \; A_4 \; \_\_$

$p = 1/5$ since 5 equally likely slots the second nonace can go,

$$\boxed{E(X) = 48(1/5)}$$

③ Sec 3.2 Tail Sum formula for expectation

Suppose X is a count $0, 1, 2, 3, \dots$

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) + \cdots$$

$$= \underbrace{1 \cdot P(X=1)}_{p_1} + 2 \cdot \underbrace{P(X=2)}_{p_2} + \cdots$$

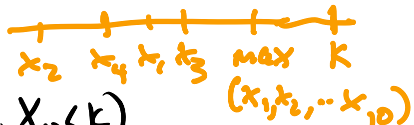
$$= \begin{array}{cccc} P(X \ge 1) & P(X \ge 2) & P(X \ge 3) & \\ p_1 & & & \\ p_2 & p_2 & & \\ p_3 & p_3 & p_3 & \\ p_4 & p_4 & p_4 & p_4 \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

$$= P(X \ge 1) + P(X \ge 2) + P(X \ge 3) + \cdots \quad \text{Tail Sum Formula}$$

This is useful when it is easy to find $P(X \ge k)$.

ex A fair die is rolled 10 times.
Let $X = \max(X_1, \dots, X_{10})$.
Find $P(X \ge k)$

$$P(X \ge k) = 1 - P(X < k)$$



$$= 1 - P(X_1 < k, X_2 < k, \dots, X_{10} < k)$$

$$= 1 - P(X_1 < k) P(X_2 < k) \cdots P(X_{10} < k)$$

$$= 1 - P(X_1 < k)^{10}$$

$$= 1 - \left(\frac{k-1}{6}\right)^{10} \quad \text{for } 1 \le k \le 6$$

use $P(X \ge k) = 0$ if $k > 6$

so $E(X) = \underset{1}{P(X \ge 1)} + \underset{1 - \left(\frac{1}{6}\right)^{10}}{P(X \ge 2)} + \cdots + \underset{1 - \left(\frac{5}{6}\right)^{10}}{P(X \ge 6)} + \underset{0}{P(X \ge 7)} + \cdots$

$$= 6 - \left(\frac{1}{6}\right)^{10} \left[1^{10} + 2^{10} + 3^{10} + 4^{10} + 5^{10}\right] = 5.82$$

ex A fair die is rolled 3 times, X_1, X_2, X_3.

Let Y be the sum of the largest 2 numbers.

Notice that $Y = X_1 + X_2 + X_3 - \min(X_1, X_2, X_3)$.

a) Find $P(\min(X_1, X_2, X_3) \geq 2)$

Picture

$$= P(X_1 \geq 2, X_2 \geq 2, X_3 \geq 2)$$

$$= P(X_1 \geq 2)^3 = \boxed{\left(\frac{5}{6}\right)^3}$$

(b) Find $E(\min(X_1, X_2, X_3))$

$$P(\min \geq 1) + P(\min \geq 2) + \dots + P(\min \geq 6)$$

$$P(X_1 \geq 1)^3 \qquad (X_1 \geq 2)^3 \qquad (X_1 \geq 6)^3$$

$$= 1^3 + \left(\frac{5}{6}\right)^3 + \left(\frac{4}{6}\right)^3 + \dots + \left(\frac{1}{6}\right)^3 = \frac{1}{6^3}\left[6^3 + 5^3 + \dots + 1^3\right]$$

(c) Find $E(Y) = E(X_1) + E(X_2) + E(X_3) - E(\min)$

$$E(X_1) = 1 \cdot \tfrac{1}{6} + 2 \cdot \tfrac{1}{6} + 3 \cdot \tfrac{1}{6} + \dots + 6 \cdot \tfrac{1}{6} = \tfrac{7}{2}$$

$$\boxed{E(Y) = 3 \cdot \frac{7}{2} - \frac{1}{6^3}\left[6^3 + 5^3 + \dots + 1^3\right]}$$

Extra Practice

. (3 pts) On a telephone wire, n birds sit arranged in a line. A noise startles them, causing each bird to look left or right at random. Calculate the expected number of birds which are not seen by an adjacent bird.

X = # birds not seen by an adjacent bird

$X = I_1 + I_2 + \dots + I_n$

$p = 1/2$

$I_1 = \begin{cases} 1 & \text{if } 1^{st} \text{ bird not seen} \\ 0 & \text{else} \end{cases}$

$p = 1/2 \cdot 1/2 = 1/4$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ bird not seen} \\ 0 & \text{else} \end{cases}$

$$\boxed{E(X) = 2 \cdot 1/2 + (n-2) \cdot 1/4}$$

ex Suppose a fair die is rolled 10 times.
Let X = number of different faces that appear in 10 rolls.

ex if roll 2, 3, 4, 2, 3, 5, 2, 3, 3, 2 then $X = 4$

$$X = I_1 + I_2 + \cdots + I_6$$

$$p = 1 - \left(\frac{5}{6}\right)^{10}$$

$$I_2 = \begin{cases} 1 & \text{if 2 appears at least once} \\ 0 & \text{else} \end{cases}$$

$$E(X) = 6 \cdot \left(1 - \left(\frac{5}{6}\right)^{10}\right)$$