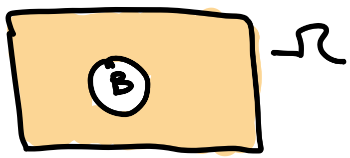
Stat 134 lec 2

Warm up 11:00 - 11:10

Prove the complement rule

$$P(B^c) = 1 - P(B)$$

Picture



Difference rule
if $B \subseteq A$
$P(AB^c) = P(A) - P(B)$

$\Omega = B \cup B^c$ disjoint union

axiom

$1 = P(\Omega) = P(B) + P(B^c)$ addition rule

$P(B^c) = 1 - P(B)$

or using difference rule:

trick: $A = \Omega$

$B^c = \Omega B^c$

$P(B^c) = P(\Omega B^c) = P(\Omega) - P(B) = 1 - P(B)$ ✓

Last time

unconditional probability

Addition rule (OR) if A, B mutually exclusive sets

$P(A \text{ or } B) = P(A) + P(B)$.

Inclusion exclusion (OR) $P(A \text{ or } B) = P(A) + P(B) - P(AB)$

Today

0. Mathematical Induction
1. Sec 1.3 Distributions
2. Sec 1.4 Conditional Probability

0. Mathematical Induction

A **proof by induction** consists of two cases. The first, the **base case** (or **basis**), proves the statement for $n = 0$ (1) without assuming any knowledge of other cases. The second case, the **induction step**, proves that *if* the statement holds for any given case $n = k$, *then* it must also hold for the next case $n = k + 1$. These two steps establish that the statement holds for every natural number n.

ex (1.3.12 in HW#1)

12. Inclusion–exclusion formula for n events. Derive the inclusion–exclusion formula for n events

$$P(\bigcup_{i=1}^{n} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} P(A_i A_j A_k) - \cdots + (-1)^{n+1} P(A_1 \ldots A_n)$$

ex $n=3$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 A_2) - P(A_1 A_3) - P(A_2 A_3) + P(A_1 A_2 A_3)$$

Lets assume the following fact from set theory:

$$\bigcup_{i=1}^{k}(A_i A_{k+1}) = \left(\bigcup_{i=1}^{k} A_i\right) A_{k+1} \quad (*)$$

$(A_1 \cup A_2) \cap A_3$

ex $\quad A_1A_3 \cup A_2A_3 = (A_1 \cup A_2)A_3 \qquad k=2$

To prove generalized inclusion exclusion.

Show base case $n=1$
then assume true for $n=k$ and show true for $n=k+1$.

To get going let $k=2$

$$P\left(\bigcup_{i=1}^{2} A_i \cup A_3\right) = P\left(\bigcup_{i=1}^{2} A_i\right) + P(A_3) - P\left(\left(\bigcup_{i=1}^{2} A_i\right) A_3\right)$$

$$= \bigcup_{i=1}^{2} A_i A_3 \text{ by } (*)$$

$$= P\left(\bigcup_{i=1}^{2} A_i\right) + P(A_3) - P\left(\bigcup_{i=1}^{2} A_i A_3\right)$$

$P\left(\bigcup_{i=1}^{2} A_i\right) = P(A_1)+P(A_2)-P(A_1A_2)$

$P\left(\bigcup_{i=1}^{2} A_i A_3\right) = P(A_1A_3)+P(A_2A_3)-P(A_1A_3A_2A_3)$

$A_1A_3A_2A_3 = A_1A_2A_3$

$$= \sum_{i=1}^{3} P(A_i) - \sum_{i<j} P(A_iA_j) + P(A_1A_2A_3)$$

For HW,
try $k=3$ and generalize.

$$P\left(\bigcup_{i=1}^{3} A_i \cup A_4\right) = P\left(\bigcup_{i=1}^{3} A_i\right) + P(A_4) - P\left(\bigcup_{i=1}^{3} A_i A_4\right)$$

① Sec 1.3 Distributions

Uniform distribution

Let $\{x_1, x_2, \ldots, x_n\}$ be a finite set.

Suppose the probability of drawing each element is equally likely (i.e each has prob $\frac{1}{n}$)

We say $\{x_1, \ldots, x_n\}$ has the uniform distribution.

We write $\text{Unif}(\{x_1, \ldots, x_n\})$.

ex $\{1, 1, 2\}$ is a finite set.

$\text{Unif}(\{1, 1, 2\})$ means 1 has probability $\frac{2}{3}$ and 2 has probability $\frac{1}{3}$.

ex Suppose[7] a[1] word[4] is[2] randomly[8] picked[6] from[4] this[4] sentance[8].

What is the distribution of the length of the word picked?

$\text{Unif}(\{7, 1, 4, 2, 8, 6, 4, 4, 8\})$

$P(x=7) = 1/9$

$P(x=1) = 1/9$

$P(x=4) = 3/9$

$\vdots$

Stat 134

1. A deck of cards is shuffled. What is the chance that the top card is the **king** of spades **or** the bottom card is the **king** of spades

   a $\frac{1}{52}+\frac{1}{52}-\frac{1}{52}\times\frac{1}{52}$
   b $\frac{1}{52}+\frac{1}{51}$
   c $\frac{1}{52}+\frac{1}{52}-\frac{1}{52}\times\frac{1}{51}$
   d none of the above — $\frac{1}{52}+\frac{1}{52}=\frac{1}{26}$

K of S on top and bottom are mutually exclusive so have addition rule

$P(\underset{top}{KS}) + P(\underset{bot}{KS})$ ← unconditional probability.

$\quad = \frac{1}{52} \qquad\quad = \frac{1}{52}$

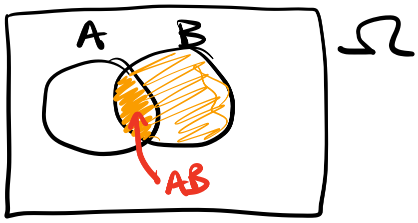
Note!
If replace card after you draw a card the two events are no longer mutually exclusive so use inclusion exclusion formula $\frac{1}{52}+\frac{1}{52}-\frac{1}{52}\cdot\frac{1}{52}$

↑ 52 not 51 since replace card.

If draw without replacement but bottom card QS then
$\frac{1}{52}+\frac{1}{52}-\frac{1}{52}\cdot\frac{1}{51}$

② <u>Sec 1.4</u> <u>Conditional Probability and Independence</u>

Let A, B be subsets of Ω (i.e events).



Baye's rule says $P(A|B) = \dfrac{P(AB)}{P(B)}$ — given

$\Leftrightarrow$ $\boxed{P(AB) = P(A|B)P(B)}$ multiplication rule. (AND)

A and B

We say A and B are <u>independent</u> iff

$P(A|B) = P(A)$

or equivalently if $P(AB) = P(A)P(B)$

ex A = last card is queen of spades
B = 1st card is king of spades } assume drawing w/o replacement

Is A and B independent?

$$P(AB) = \underset{\overset{\shortparallel}{1/52}}{P(B)}\,\underset{\overset{\shortparallel}{1/51}}{P(A|B)} \neq \underset{\overset{\shortparallel}{1/52}}{P(B)}\,\underset{\overset{\shortparallel}{1/52}}{P(A)}$$

$\Rightarrow$ A, B <u>dependent</u>

ex

(10 pts) An airport bus drops off 35 passengers at 7 stops. Each passenger is equally likely to get off at any stop, and passengers act independently of one another. The bus makes a stop only if someone wants to get off. Find the probability that the bus drops off passengers at every stop.

Let B_i = event drop off at least one person at stop i.

$P(B_1 B_2 \cdots B_7)$

$P(B_1) = 1 - \underset{\left(\frac{6}{7}\right)^{35}}{\underset{\parallel}{P(B_1^c)}}$

De morgan's rule $(AB)^c = A^c \cup B^c$

$P(B_1B_2) = 1 - P((B_1B_2)^c) = 1 - \underbrace{P(B_1^c \cup B_2^c)}$

$P(B_1^c) + P(B_2^c) - P(B_1^c B_2^c)$

$\left(\frac{6}{7}\right)^{35} \quad \left(\frac{6}{7}\right)^{35} \quad \left(\frac{5}{7}\right)^{35}$

$P(B_1B_2B_3) = 1 - P((B_1B_2B_3)^c)$

$= 1 - P(B_1^c \cup B_2^c \cup B_3^c)$

$= 1 - \sum_{i=1}^{3} P(B_i^c) + \sum_{i<j} P(B_i^c B_j^c) - P(B_1^c B_2^c B_3^c)$

$\binom{3}{1}\left(\frac{6}{7}\right)^{35} \quad \binom{3}{2}\left(\frac{5}{7}\right)^{35} \quad \binom{3}{3}\left(\frac{4}{7}\right)^{35}$

$\binom{3}{1} = 3 \quad \binom{3}{2} = 3 \quad \binom{3}{3} = 1$

$P(B_1B_2\cdots B_7) = 1 - \binom{7}{1}\left(\frac{6}{7}\right)^{35} + \binom{7}{2}\left(\frac{5}{7}\right)^{35} - \cdots - \binom{7}{7}\left(\frac{7-7}{7}\right)^{35}$

$= \boxed{\sum_{j=0}^{7} (-1)^j \binom{7}{j}\left(\frac{7-j}{7}\right)^{35}}$

Inclusion–exclusion formula for n events. Derive the inclusion–exclusion formula for n events

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) - \cdots + (-1)^{n+1} P(A_1 \ldots A_n)$$