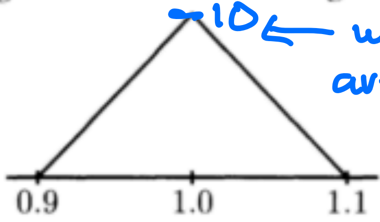
Stat 134 lec 20

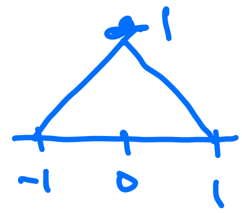
warmup

Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



← we know this since area =

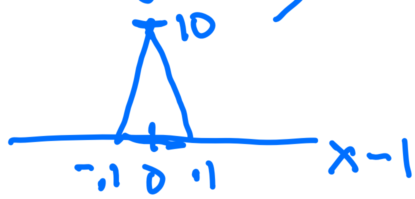
X-1



You should change the scale of X= the length of rods to:

10(X-1)

- a: X-1
- b: .1(X-1)
- c: 10X-1
- ~~d: none of the above~~



Last time sec 4.1 Continuous distributions

A continuous RV X, has a prob density function, $f(x)$, where $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

$P(X=a) = \int_a^a f(x)dx = 0$ so $P(X \ge a) = P(X > a)$.

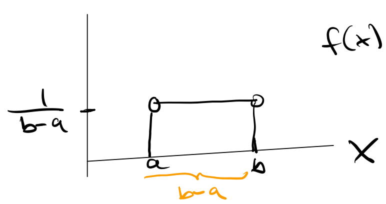
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx$$

A change of scale is a transformation $Y = m + nX$, of X. The purpose is that it makes it easier to calculate $E(X)$ and $Var(X)$.

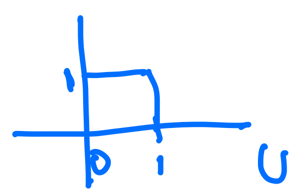
constants.

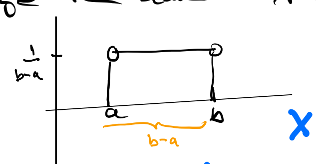
let $X \sim Unif(a,b)$

$$f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{else} \end{cases}$$



a) You should change the scale of X to? $\boxed{U = \frac{1}{b-a}(X-a)}$





b) Find $E(X)$

$X = (b-a)U + a$

$E(X) = (b-a)E(U) + a = (b-a)\cdot\frac{1}{2} + a$

$= \boxed{\frac{a+b}{2}}$

c) Find $Var(X)$.

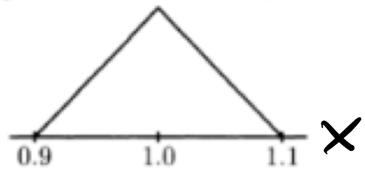
$X = (b-a)U + a$

$Var(X) = Var((b-a)U + a) = (b-a)^2 \underbrace{Var(U)}_{= 1/12} = \boxed{\frac{(b-a)^2}{12}}$

Today

1. sec 4.1 change of scale calculations
2. briefly sec 4.5 Cummulative Distribution Function (CDF)
3. sec 4.2 Exponential Distribution.

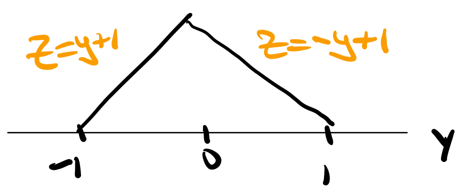
Suppose a manufacturing process designed to produce rods of length 1 inch exactly, in fact produces rods with length distributed according to the density graphed below.



Find the variance of the length of the rods.

$Y = 10(X-1)$ change of scale. easier to find.

$\text{Var}(Y) = 100\,\text{Var}(X) \Rightarrow \text{Var}(X) = \dfrac{\text{Var}(Y)}{100}$



Find the density of Y: ✓

Find Var (X)

$\text{Var}(Y) = E(Y^2) - \big(\underbrace{E(Y)}_{=0}\big)^2$

$E(Y^2) = \displaystyle\int_{-1}^{0} y^2 \cdot (y+1)\,dy + \int_{0}^{1} y^2(-y+1)\,dy$

$= \boxed{1/6}$

$\text{Var}(Y) = 1/6$

$\text{Var}(X) = \dfrac{\text{Var}(Y)}{100} = \boxed{\dfrac{1}{600}}$

② briefly sec 4.5 The Cummulative Distribution Function (CDF)

Let X be a continuous RV

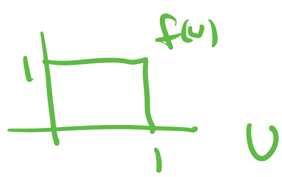
$F(x) = P(X \leq x)$ — a number between 0 and 1

If $f(x)$ is a density of X,

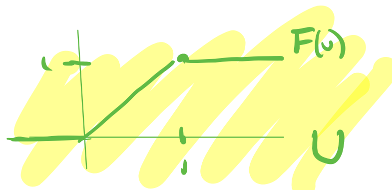
$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

ex $U \sim \text{Unif}(0,1)$

$$f(u) = \begin{cases} 1 & 0 < u < 1 \\ 0 & \text{else} \end{cases}$$



$$F(u) = \int_0^u 1\,dx = u$$



$$\boxed{F(u) = \begin{cases} 0 & -\infty < u \leq 0 \\ u & 0 \leq u \leq 1 \\ 1 & u \geq 1 \end{cases}}$$

By FTC, $F'(x) = f(x)$

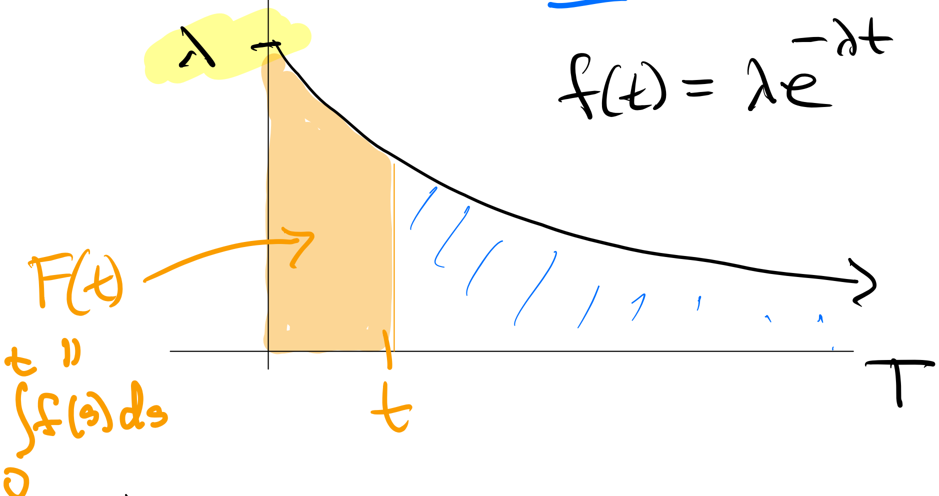
Consequently a density function and cdf are equivalent descriptions of a RV.

③ Sec 4.2   Exponential distribution

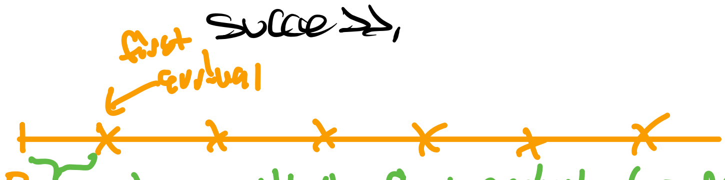
Def^n A random time T has exponential distribution with rate $\lambda > 0$.

$T \sim \text{Exp}(\lambda)$, if T has density
$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$

Fact $E(T) = \frac{1}{\lambda}$



ex T = time until your first success where λ = rate of success,



~ time until the first arrival of a poisson process with rate λ.

ex $T=$ time until a lightbulb burns out

CDF and survival function

$T \sim \text{Exp}(\lambda) \quad f(t)=\lambda e^{-\lambda t}$

Compute the CDF of T.

$$F(t)=P(T\le t) = \int_{-\infty}^{t} f(s)\,ds$$

$$= \underbrace{\int_{-\infty}^{0} f(s)\,ds}_{=0} + \int_{0}^{t} f(s)\,ds = \int_{0}^{t} \lambda e^{-\lambda s}\,ds$$

$$= \frac{\lambda e^{-\lambda s}}{-\lambda}\Bigg|_{0}^{t} = -e^{-\lambda t}+1$$

$$= \boxed{1-e^{-\lambda t}}$$

$P(T>t)=e^{-\lambda t}$ is called the <u>survival function</u>

$T\sim \text{Exp}(\lambda)$ iff $P(T>t)=e^{-\lambda t}$

since $\underset{=1-P(T>t)}{F(t)}$ and $f(t)$ both define distribution.

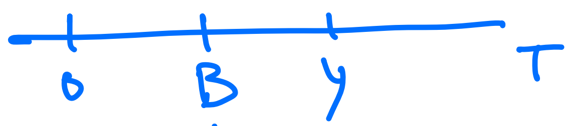
ex (competing exponentials)

GSI Brian and Yiming are each helping a student. Brian and Yiming see students at a rate λ_B and λ_Y students per hour respectively.

let B = wait time for Brian $\sim Exp(\lambda_B)$

Y = wait time for Yiming $\sim Exp(\lambda_Y)$

What distribution is $X = \min(B, Y)$?



$$P(X > x) = P(B > x, Y > x)$$

$$= P(B > x)P(Y > x) = e^{-(\lambda_B + \lambda_Y)x}$$

$$\qquad e^{-\lambda_B x} \quad e^{-\lambda_Y x}$$

$$\Rightarrow \boxed{X \sim Exp(\lambda_B + \lambda_Y)}$$

Memoryless property of Exponential

survival function

$T \sim Exp(\lambda)$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ $\quad$ or $\quad P(T > t) = e^{-\lambda t}$

T = time until 1^{st} success (arrival)

Memoryless Property:

$$P(T > t+s \mid T > s) = P(T > t)$$

Pf/

Bayes rule

$$P(T > t+s \mid T > s) = \frac{P(T > t+s, \cancel{T > s})}{P(T > s)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda(s)}} = e^{-\lambda t} = P(T > t)$$

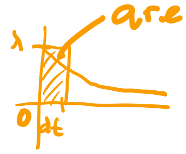
Comments:

① If T = lifetime of a lightbulb, we assume so long as a lightbulb is still functioning it is as good as new

another way to write the Memoryless Property

② $P(T \in dt \mid T > t) = P(0 < T < 0 + dt) \approx \lambda dt$

Picture



area $\lambda dt \approx P(0 < T < 0 + dt)$

$$\Rightarrow \lambda = \frac{P(T \in dt \mid T > t)}{dt}$$

the instantaneous rate of success is the same for all