Stat 134 lec 21

Warmup: 10:00-10:10

Recall $P(T>k) = e^{-\lambda k}$

Let $T \sim Exp(\lambda)$

a) Find $P(T>5) = e^{-5\lambda}$

b) Find $P(T>\underset{5+8}{\underset{\parallel}{13}} \mid T>8) = e^{-5\lambda}$

memoryless property of exponential,

or Bayes rule gives

$$\frac{P(T>13)}{P(T>8)} = e^{-5\lambda} \checkmark$$

last time

Sec 4.5 Cummulative Distribution Function (CDF)

The CDF of a RV X is $P(X \le x)$. The survival function is $P(X > x)$.

The CDF or survival function uniquely determine a distribution.

sec 4.2 exponential distribution.

survival function

$T \sim Exp(\lambda)$ $f(t) = \begin{cases} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & \text{else} \end{cases}$ or $P(T > t) = e^{-\lambda t}$

T = time until 1^{st} success (arrival)

Memoryless property of the geometric distribution and exponential distribution

memoryless property

$P(X > k+j \mid X > j) = P(X > k)$

proved in appendix of notes

Thm The Geometric distribution is the only discrete distribution with values 1,2,3... having the **memoryless property**

Only 2 distributions are memoryless:

For discrete $(X = 1,2,3,\dots)$ − Geometric

For continuous $(T > 0)$ − Exponential ← Proof is similar to geom.

Today sec 4.2

1. Expectation and Variance of $Exp(\lambda)$.
2. Gamma distribution.

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Sec 4.2 Expectation and Variance of Exp(λ)

$T \sim \text{Exp}(\lambda)$

see end of lec notes.

$$E(T) = \int_0^\infty t f(t)\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt = \boxed{\frac{1}{\lambda}}$$

see end of lec notes.

$$E(T^2) = \int_0^\infty t^2 f(t)\,dt = \lambda \int_0^\infty t^2 e^{-\lambda t}\,dt = \frac{2}{\lambda^2}$$

$$\text{var}(T) = E(T^2) - E(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \boxed{\frac{1}{\lambda^2}}$$

Interpretation of λ:

$\lambda = \frac{1}{E(T)}$ (one over average arrival time)

This makes sense since the instantaneous rate of success is constant by the memoryless property of Exp(λ).

So if it takes on avg 5 min to have a success then $\lambda = 1/5$.

ex Let T = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find $P(T>3)$

$E(T) = 2 \Rightarrow \lambda = \frac{1}{2}$

$T \sim \text{Exp}(1/2)$

$P(T>3) = e^{-\frac{1}{2}\cdot 3} = \boxed{e^{-3/2}}$

ex

A family is getting ready for their trip to Yosemite. Each person is in their room, packing their bags. For each person, the time it takes them to pack their bag is exponentially distributed and independent of the time it takes any other person. On average, it takes each parent 1 hour and each child 2 hours to get ready. In a family with 2 parents and 4 children, what is the probability that it takes the family more than 2 hours to get ready?

$P_1, P_2, C_3, C_4, C_5, C_6$

$P_1, P_2 \sim Exp(1)$

$C_3, C_4, C_5, C_6 \sim Exp(1/2)$

$M = max(P_1, P_2, C_3, C_4, C_5, C_6)$

Find $P(M > 2)$

$$P(M \le 2) = P(P_1 \le 2)P(P_2 \le 2)P(C_3 \le 2) \cdots P(C_6 \le 2)$$
$$= (1-e^{-1 \cdot 2})^2 (1-e^{-1/2 \cdot 2})^4$$

$$P(M > 2) = 1 - P(M \le 2)$$
$$= \boxed{1 - (1-e^{-1 \cdot 2})^2 (1-e^{-1/2 \cdot 2})^4}$$

② Gamma Distribution

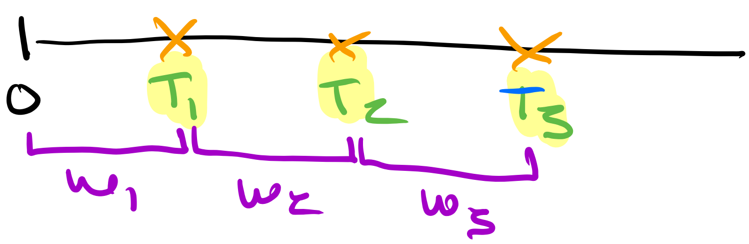
A gamma distribution, $T_r \sim \text{Gamma}(r, \lambda)$, $\lambda > 0$, is a sum of r iid $\text{Exp}(\lambda)$.

$$T_r = W_1 + W_2 + \dots + W_r \;, \quad W_i \sim \text{Exp}(\lambda)$$

ex T_r = time of the r^{th} arrival of a Poisson Process $(\text{Pois}(\lambda t))$

ex T_r = time when the r^{th} lightbulb dies.

Picture



$$\left.\begin{aligned} T_1 &\sim \text{Gamma}(1, \lambda) \\ T_2 &\sim \text{Gamma}(2, \lambda) \\ T_3 &\sim \text{Gamma}(3, \lambda) \end{aligned}\right\}$$

dependent but W_1, W_2, W_3 are independent, $\text{Exp}(\lambda)$

$$T_1 = W_1$$
$$T_2 = W_1 + W_2$$
$$T_3 = W_1 + W_2 + W_3$$

Review Poisson Process (a.k.a PRS)

$N_t \sim Pois(\lambda t)$

N_t = # arrivals in time t where the rate of arrivals is λ.

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

ex Let T_1 = waiting time (min) until a blue Honda enters a toll gate starting at noon. The waiting time is on average 2 minutes. Find the probability 3 blue Hondas arrive in 8 minutes.

$t = 8$

$\lambda = 1/2$

$N_8 \sim Pois(\underbrace{\tfrac{1}{2}\cdot 8}_{\mu = 4})$

$$P(N_8 = 3) = \boxed{\frac{e^{-4}4^3}{3!}}$$

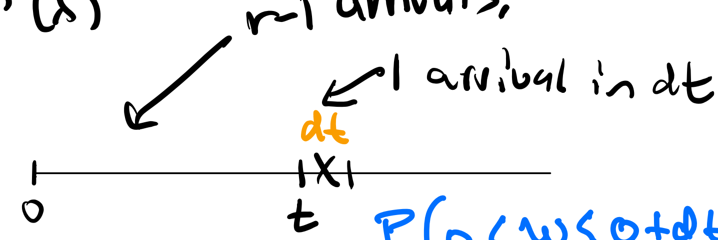
Back to gamma distribution: $T_r \sim \text{Gamma}(r, \lambda)$

Let $N_t \sim \text{Pois}(\lambda t)$

$W \sim \text{Exp}(\lambda)$

$\approx f(t)\,dt$

$P(T_r \in dt)$ means



$$P(0 < W < 0 + dt) \approx \lambda\, dt$$

so $$P(T_r \in dt) = P(N_t = r-1)\,P(W \in dt \mid W > t)$$

$$\approx \frac{e^{-\lambda t}(\lambda t)^{r-1}}{(r-1)!} \cdot \lambda\, dt$$

$$= \underbrace{\frac{1}{(r-1)!}\lambda^r t^{r-1} e^{-\lambda t}}_{f(t)}\, dt$$

<u>Gamma function $\Gamma(r)$, $r > 0$</u>

If $r \in \mathbb{Z}^+$ define $\Gamma(r) = (r-1)!$ where $\Gamma(r)$ is called the <u>gamma function</u>

For $\lambda = 1$, $f(t) = \underbrace{\frac{1}{\Gamma(r)}}_{\text{const}} \underbrace{t^{r-1} e^{-t}}_{\text{variable}}$

and $\int_0^\infty f(t)\,dt = 1 \Rightarrow$

$$\frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-t}\,dt = 1$$

$$\Rightarrow \Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\,dt$$

This allows us to define

$T \sim \text{Gamma}(r, \lambda)$ for $r > 0$ as

having density $f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$

$t \geq 0$

ex Let $T_r \sim \text{Gamma}(r=4, \lambda=2)$

Find $f(t)$

$$f(t) = \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}$$

$$= \frac{1}{3!} 2^4 t^3 e^{-2t} = \boxed{\frac{8}{3} t^3 e^{-2t}}$$

The formula $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t} \, dt$.

is the definition of the gamma function.

You can show that $\Gamma(r) = (r-1)!$

for $r \in \mathbb{Z}^+$

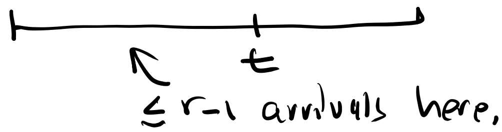
Now, $T_r \sim \text{Gamma}(r, \lambda)$ has density

$$f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else} \end{cases}$$

Let $T_r \sim \text{Gamma}(r, \lambda)$

$$P(T_r > t) = \int_t^{\infty} \frac{1}{\Gamma(r)} \lambda^r s^{r-1} e^{-\lambda s}\, ds$$

$N_t \sim \text{Pois}(\lambda t)$



$$P(T_r > t) = P(N_t < r)$$

$$= \sum_{i=0}^{r-1} P(N_t = i)$$

$$= \boxed{\sum_{i=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^i}{i!}}$$

Ex let $T_4 \sim \text{Gamma}(r=4, \lambda=2)$

Find $P(T_4 > 7)$

$$P(T_4 > 7) = P(N_7 < 4) = P(N_7 \leq 3)$$

$$N_7 \sim \text{Pois}(\underset{\underset{14}{\parallel}}{2 \cdot 7})$$

$$= e^{-14}\left(1 + \frac{14}{1!} + \frac{14^2}{2!} + \frac{14^3}{3!}\right)$$

Appendix

First we show that $P(X > k) = q^k$ is an equivalent def^n of Geometric on $1, 2, 3, \dots$

let $X \sim \text{Geom}(p)$ where

$X = \#$ trials until your first success.

$$P(X = k) = q^{k-1} p$$

recall
$$P(X > k) = P(X = k+1) + P(X = k+2) + \dots$$
$$= q^k p + q^{k+1} p + \dots$$
$$= q^k p \underbrace{(1 + q + q^2 + \dots)}_{= \frac{1}{1-q}}$$
$$= q^k.$$

Since
$$P(X = k) = \underbrace{P(X > k-1)}_{= q^{k-1}} - \underbrace{P(X > k)}_{= q^k}$$
$$= q^{k-1}(1-q) = q^{k-1} p$$

$P(X > k) = q^k \Rightarrow X$ is Geom(p).

$\Rightarrow X \sim \text{Geom}(p)$ iff $P(X > k) = q^k$

Thm The Geometric distribution is the only discrete distribution with values $1, 2, 3, \ldots$ having the

memoryless property

$$P(X > k+j \mid X > j) = P(X > k)$$

Proof/

Let $X \sim \text{Geom}(p)$, $X = 1, 2, 3, \ldots$

$$P(X > k+j \mid X > j) = \frac{P(X > k+j, X > j)}{P(X > j)}$$

$$= \frac{P(X > k+j)}{P(X > j)}$$

$$= \frac{q^{k+j}}{q^{j}} = q^{k}$$

$= P(X > k)$ ✓

Conversely,

For positive integers k, j

Suppose $P(X > k+j \mid X > j) = P(X > k)$

$$\| $$

$$\frac{P(X > k+j)}{P(X > j)}$$

$\iff P(X > k+j) = P(X > j)P(X > k)$

let $q = P(X > 1)$

Show that $P(X > j) = q^j$

for $j = 1, 2, 3, \ldots$ by induction.

Base case :

$P(X>1) = q^1$ ✓

Assume $P(X>j-1) = q^{j-1}$

$$P(X>j) = P(X>j-1+1)$$

$$= \underbrace{P(X>j-1)}_{= q^{j-1}} \underbrace{P(X>1)}_{= q} = q^j$$

$\Rightarrow X \sim \text{Geom}(p)$.

$\square$

Appendix

Expectation and Variance of exponential

let $T \sim Exp(\lambda)$

$f(t) = \lambda e^{-\lambda t}$ density

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt$$

I recommend using the <u>tabular method</u> for integration by parts. This works well when the function you are integrating is the product of two expressions, where the n^{th} derivative of one expression is zero.

ex $t^4 e^{3t}$ ✓ good

ex $\sin t \, e^{3t}$ ✗ bad.

To find $\lambda \int_0^\infty t e^{-3t}$:

$\frac{d}{dt}$		$\int$
t	$+$	$e^{-\lambda t}$
1	$-$	$\frac{-e^{-\lambda t}}{\lambda}$
0		$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(T) = \lambda \int_0^\infty t e^{-\lambda t} dt = \lambda \left(t\left(-\frac{e^{-\lambda t}}{\lambda}\right) - 1 \cdot \frac{e^{-\lambda t}}{\lambda^2} \right) \Bigg|_0^\infty$$

$$= 0 - \left(0 - \frac{1}{\lambda}\right)$$

$$= \boxed{\frac{1}{\lambda}}$$

Next find $Var(T) = E(T^2) - E(T)^2$:

$$E(T^2) = \lambda \int_0^\infty t^2 e^{-\lambda t} dt$$

$\frac{d}{dt}$		$\int$
t^2	$+$	$e^{-\lambda t}$
$2t$	$-$	$-\frac{1}{\lambda} e^{-\lambda t}$
2	$+$	$\frac{1}{\lambda^2} e^{-\lambda t}$
0		$-\frac{1}{\lambda^3} e^{-\lambda t}$

$$E(T^2) = \lambda \left(t^2\left(-\frac{1}{\lambda} e^{-\lambda t}\right) - 2t\left(\frac{1}{\lambda^2} e^{-\lambda t}\right) + 2\left(-\frac{1}{\lambda^3} e^{-\lambda t}\right) \right) \Bigg|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\Rightarrow Var(T) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$