Stat 134 <u>lec 23</u>

Warmup: 10:00–10:10

Let $X \sim Ber(p)$, $X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{with prob } 1-p \end{cases}$

a) Find $E(X) = 1 \cdot p + 0 \cdot (1-p) = p$

$E(X^2) = 1^2 \cdot p + 0^2 (1-p) = p$

$\vdots$

$E(X^k) = p$

called moments of X

think of e^{tX} as a function $g(X)$ of $X \sim Ber(p)$

b) Find $E(e^{tX})$, $t \in \mathbb{R}$

$e^{t \cdot 1} \cdot p + e^{t \cdot 0}(1-p)$

$= e^{t \cdot 1} \cdot p + 1 - p$

$= pe^t + 1 - p = \boxed{1 + p(e^t - 1)}$ ← for all $t \in \mathbb{R}$

$$\frac{d}{dt}\left(E(e^{tX})\right)\Big|_{t=0} = \frac{d}{dt}\left(1 + p(e^t - 1)\right)\Big|_{t=0} = pe^t\Big|_{t=0} = p$$

$$\frac{d^2}{dt^2}\left(E(e^{tX})\right)\Big|_{t=0} = \frac{d}{dt}\left(pe^t\right)\Big|_{t=0} = p$$

$\vdots$

$$\frac{d^k}{dt^k}\left(E(e^{tX})\right)\Big|_{t=0} = p$$

To find the moments of X we take the derivatives of E^{tX} and evaluate at $t=0$.

For X a RV, $M_X(t) = E(e^{tX})$ is called the moment generating function (MGF) of X.

Special lecture

Moment Generating Function of X

Not in book.

Next time Finish MGF, Sec 4.4, start Sec 4.5

Moment Generating Function (MGF) of X

The k^{th} moment of a RV X is the number

$E(X^k)$ defined for $k = 0, 1, 2, 3, \ldots$

$E(X^0) = E(1) = 1$

$E(X)$

$E(X^2)$

moments describe your distribution
ex 1^{st} moment is mean
2^{nd} moment relates to variance
3^{rd} moment relates to how skewed the distribution is.

Note $Var(X) = E(X^2) - E(X)^2$

Computing the moments of a RV is sometimes hard, because it involves computing complicated integrals or sums

$$E(X) = \begin{cases} \int_{-\infty}^{\infty} x f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{-\infty}^{\infty} x P(x) & \text{if } X \text{ discrete} \end{cases}$$

The MGF allows one to compute the moments by computing derivatives, which is easier.

Recall $E(g(x)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

We define the MGF of X to be

$$M_X(t) = E(e^{tX}) = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)\,dx & \text{if } X \text{ continuous} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x) & \text{if } X \text{ discrete} \end{cases}$$

$M_X(t)$ is sometimes written $\psi_X(t)$ in HW 9

In the warmup we saw for $X \sim Ber(p)$

k^{th} derivative evaluated at $t=0$

$$M_X^{(k)}(t)\Big|_{t=0} = E(X^k)$$

Let's show that is true for most RV X.

Recall the Taylor series for e^y:

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots$$

let $t \in \mathbb{R}$, X RV.

You can do this ~~for~~ the RV tX

$$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} +$$

$$E(e^{tX}) = E(1) + E(tX) + E\left(\frac{(tX)^2}{2!}\right) + \cdots$$

$M_X(t)$

$$= E(1) + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots$$

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = E(X)$$

$$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = E(X^2)$$

more generally,

$$\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \frac{d^k}{dt^k} E(e^{tX})\Big|_{t=0}$$

Lets not worry about why this step is true. See Leibniz rule.

$$\overset{!!!}{=} E\left(\frac{d^k}{dt^k} e^{tX}\Big|_{t=0}\right)$$

$$= E(X^k)$$

Summary, the MGF $M_X(t) = E(e^{tX})$ contains info about all of the moments of X. By taking the k^{th} derivative and evaluating at 0 you get the k^{th} moment.

Thm If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

Note An MGF doesn't always exist in an interval around zero. (see appendix for an example)

ex let $X \sim \text{Gamma}(r, \lambda)$

$$f(x) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r x^{r-1} e^{-\lambda x}, & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Recall $\Gamma(r) = \int_0^\infty u^{r-1} e^{-u} du$

Find $M_X(t)$.

Step 1 Write $M_X(t)$ as an integral

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} \left(\frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \right) dx$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{(t-\lambda)x} dx \quad \text{for } t < \lambda$$

Step 3 Solve the integral

Hint:

make a u substitution

let

$$u = \underbrace{-(t-\lambda)}_{>0} x$$

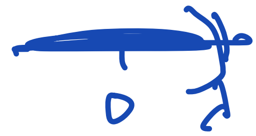
so $x = -\dfrac{u}{t-\lambda} = \dfrac{u}{\lambda - t}$, $dx = \dfrac{1}{\lambda - t} du$

$$\frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{(t-\lambda)x} dx =$$

$$= \frac{\lambda^r}{\Gamma(r)} \int_{u=0}^{u=\infty} \frac{u^{r-1}}{(\lambda - t)^{r-1}} e^{-u} \cdot \frac{1}{\lambda - t} du$$

$$= \frac{\lambda^r}{\cancel{\Gamma(r)}} \cdot \frac{1}{(\lambda - t)^r} \underbrace{\int_0^\infty u^{r-1} e^{-u}\, du}_{= \Gamma(r)}$$

$$= \frac{\lambda^r}{(\lambda - t)^r} = \boxed{\left(\frac{\lambda}{\lambda - t}\right)^r \text{ for } t < \lambda}$$



Recall If a MGF exists in an interval around zero, $M^{(r)}(t)\Big|_{t=0} = E(X^r)$

ex

Let $X \sim \text{Gamma}(r, \lambda)$

Find $\frac{d}{dt} M_X(t)\Big|_{t=0}$

Recall $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

$$M_X(t) = \lambda^r (\lambda - t)^{-r}$$

$$M'(t) = \lambda^r (-r)(\lambda - t)^{-r-1}(-1)$$

X

$$= \frac{r\lambda^r}{(\lambda-t)^{r+1}}$$

$$m'_X(0) = \boxed{\frac{r}{\lambda}}$$ ← this is $E(X)$!

Similarly,

Find: $\frac{d^2}{dt^2} m_X(t) \Big|_{t=0}$

$$m''(t) = r\lambda^r(-r-1)(\lambda-t)^{-r-2}(-1)$$

$$= r(r+1)\lambda^r \frac{1}{(\lambda-t)^{r+2}}$$

$$\Rightarrow m''(0) = r(r+1)\lambda^r \frac{1}{\lambda^{r+2}} = \frac{r(r+1)}{\lambda^2}$$

$$\Rightarrow \boxed{E(X^2) = \frac{r(r+1)}{\lambda^2}}$$

$$\Rightarrow \mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{r(r+1)}{\lambda^2} - \frac{r^2}{\lambda^2}$$

$$= \frac{r^2}{\lambda^2} + \frac{r}{\lambda^2} - \frac{r^2}{\lambda^2} = \boxed{\frac{r}{\lambda^2}}$$

ex: A RV X ~~takes~~ takes values $1, 2, 3$ with prob $1/2, 1/3, 1/6$.

Find $M_X(t)$.

$$E(e^{tX}) = \sum_{x=1}^{3} e^{tx} p(x)$$

$$= e^{\cancel{t\cdot 1}} \cdot \frac{1}{2} + e^{2t} \cdot \frac{1}{3} + e^{3t} \cdot 1/6$$

for all t

ex $X \sim Geom(\frac{1}{3})$

$P(X=x) = (\frac{2}{3})^{x-1}(\frac{1}{3})$ $\quad x=1,2,3,\dots$

Find $M_X(t) = E(e^{tX})$

$$= \sum_{x=1}^{\infty} e^{tx} (2/3)^{x-1} (1/3)$$

$$= \sum_{x=1}^{\infty} e^{t} \cdot (e^{t})^{x-1} [2/3]^{x-1} (1/3)$$

$$= \frac{1}{3} e^{t} \sum_{x=1}^{\infty} (e^{t} \cdot 2/3)^{x-1} = 1 + e^{t}\frac{2}{3} + (e^{t}2/3)^{2} + \cdots$$

$$= \frac{1}{1-(e^{t}\frac{2}{3})} \quad \text{if } e^{t}\frac{2}{3} < 1$$

or $t < \log(\frac{3}{2})$

$$\Rightarrow \boxed{M_X(t) = \frac{1}{3} e^{t} \cdot \frac{1}{1-e^{t}\cdot\frac{2}{3}} \quad \text{if } t < \log(3/2)}$$

Recall from Calculus

$$e^{x} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$ Taylor for all $x \in \mathbb{R}$

Ex $X \sim \text{Pois}(\mu)$, $P(X=k) = \frac{\mu^k e^{-\mu}}{k!}$

a) Find $M_X(t) = E(e^{tX})$

b) Find $E(X)$

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \frac{\mu^k e^{-\mu}}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu e^t)^k}{k!}$$

$= e^{\mu e^t}$ for all t

$$= e^{-\mu} \cdot e^{\mu e^t} = \boxed{e^{\mu(e^t - 1)} \text{ for all } t}$$

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = e^{\mu(e^t-1)} \cdot \mu e^t \Big|_{t=0} = \mu$$ ✓

Appendix:

Let X be a discrete RV with probability mass function $P(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, \dots \\ 0 & \text{else} \end{cases}$

The MGF, $M_X(t)$, only exists at $t \leq 0$, and hence doesn't exist on an interval around zero.

pf/

It is known that the series

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ converges to $\pi^2/6$.

Then $P(x) = \begin{cases} \frac{6}{\pi^2 x^2} & x = 1, 2, 3, \dots \\ 0 & \text{else.} \end{cases}$

is the prob function of a RV X,

$$M_X(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=1}^{\infty} \frac{6e^{tx}}{\pi^2 x^2}$$

The ratio test can be used to show this diverges if $t > 0$. Hence this RV only has an MGF at $t \leq 0$ and is not differentiable at zero. $\square$