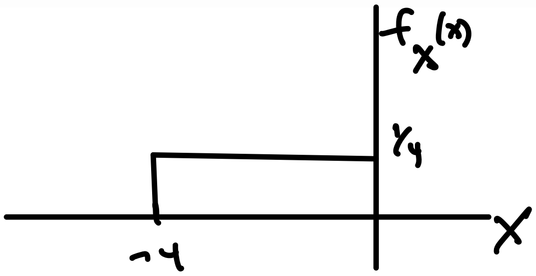
Stat 134 lec 24

warmup 10:00 - 10:10

Suppose X has uniform (-4,0) distribution. Find the density of X^2.



Change of variable formula:

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}$$ evaluated at $x = g^{-1}(y)$

1) Find $g(x) = x^2$

2) Find $g'(x) = 2x$

3) Find $x = g^{-1}(y)$ $\pm\sqrt{y}$ for neg x only have $-\sqrt{y}$

4) Find $f_X(x)$ $\frac{1}{4} \cdot 1_{x \in (-4,0)}$

5) Find $f_Y(y)$

$$\left. \frac{\frac{1}{4} \cdot 1_{x \in (-4,0)}}{|2x|} \right|_{x=-\sqrt{y}}$$

$$= \frac{\frac{1}{4} 1_{-\sqrt{y} \in (-4,0)}}{2\sqrt{y}} = \frac{\frac{1}{4} \cdot 1_{y \in (0,16)}}{2\sqrt{y}}$$

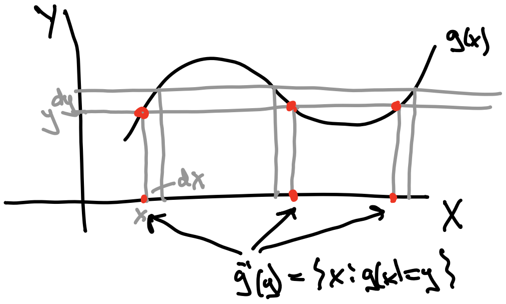
$$= \boxed{\frac{1}{8\sqrt{y}} 1_{y \in (0,16)}}$$

Announcement: Q3 Wed after break
covers Sec 4.1, 4.2, 4.4, 4.5, MGF

Last 2 times

① sec 4.4 Change of Variable rule

many to one g:



$$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$$

$$f_Y(y) = f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy} + f_X(x_3)\frac{dx_3}{dy}$$

$$= \frac{f_X(x_1)}{g'(x_1)} + \frac{f_X(x_2)}{|g'(x_2)|} + \frac{f_X(x_3)}{g'(x_3)}$$

$P(x \in dx_2) \geq 0$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{g'(x)}$$

② MGF (not in book)

$M_X(t) = E(e^{tX})$

Thm If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

ex A RV X takes values 1, 2, 3 w/ probabilities $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ respectively.

Find E(X) using MGF of X.

$M_X(t) = E(e^{tX}) = \frac{1}{2} \cdot e^t + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$

$E(X) = M_X'(0) = \frac{1}{2} + \frac{2}{3} + \frac{3}{6}$ ← confirm using def^n of E(X)

Today

① Key properties of MGF

② Recognizing a distribution from the variable part of its density.

① Key Properties of MGF

ⓐ If an MGF exists in an interval containing zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

last time

ⓑ If X and Y are independent RVs,

$M_{X+Y}(t) = M_X(t)M_Y(t)$.

Proved in MGF HW.

ⓒ If $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$ (i.e X and Y have the same distribution).

Skip proof — we can invert a MGF to get the CDF.

$= E(e^{tX})$

ex If $M_X(t) = \frac{1}{2}e^{1t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$,

e^{xt} tells us the value of X and the associated coefficients tell us the probabilities (i.e X = 1, 2, 3 w/ prob ½, ⅓, ⅙.)

so MGF ⇒ distribution of X when X has finite # values.

Property ⓐ is useful to find E(X), Var(X),

Property ⓑ and ⓒ allow us to prove for example that sum of independent Poisson is Poisson.

ex

$$\left.\begin{array}{l} X_1 \sim \text{Pois}(\mu_1) \\ X_2 \sim \text{Pois}(\mu_2) \end{array}\right\} \text{ independent.}$$

show that $X_1 + X_2 \sim \text{Pois}(\mu_1+\mu_2)$.

$$M_{X_1}(t) = e^{\mu_1(e^t-1)} \quad \text{for all } t$$

$$M_{X_2}(t) = e^{\mu_2(e^t-1)} \quad \text{for all } t$$

$$M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t) = \boxed{e^{(\mu_1+\mu_2)(e^t-1)}}$$

MGF of Pois $(\mu_1+\mu_2)$ for all t.

$\Rightarrow X_1+X_2 \sim \text{Pois}(\mu_1+\mu_2)$

ex Let X be a RV and a a constant.

Show that $M_{aX}(t) = M_X(at) \leftarrow E(e^{Xat})$

hint $M_{aX}(t) = E(e^{aXt})$

$$= E(e^{Xat})$$

$$= M_X(at)$$

For $X \sim \text{Gamma}(r, \lambda)$

recall $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

ex Let $X \sim \text{Exp}(\lambda)$ and $a > 0$.
Show that $Y = aX$ is also exponential,
and specify the new parameter.

$M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^1$ for $s < \lambda$ since $X \sim \text{Gamma}(1, \lambda)$

$Y = aX$

$$M_{aX}(t) = M_X(at) = \left(\frac{\lambda}{\lambda - at}\right)^1 \quad \text{for } at < \lambda$$

$$= \left(\frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t}\right)^1 \quad \text{for } t < \underset{>0}{\boxed{\frac{\lambda}{a}}}$$

$$\Rightarrow \boxed{Y \sim \text{Exp}\left(\frac{\lambda}{a}\right)}$$

②

Recognizing a distribution from the variable part of its density.

A density can be written as

$$f(t) = \underset{\uparrow\ \text{constant}}{c}\ \underset{\nwarrow\ \text{variable part.}}{h(t)}$$

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c\int_{-\infty}^{\infty} h(t)\,dt \Rightarrow c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So you can figure out the density from its variable part.

List of densities. Please circle their variable parts:

ex: $T \sim \text{Gamma}(r, \lambda)$ $\quad f(t) = \frac{1}{\Gamma(r)}\lambda^{r}\boxed{t^{r-1}e^{-\lambda t}}, \ t > 0$

$T \sim \text{Normal}(\mu, \sigma^2)$ $\quad f(t) = \frac{1}{\sqrt{2\pi}\,\sigma}\boxed{e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}}$

$T \sim \text{Unif}(a, b)$ $\quad f(t) = \frac{1}{b-a}\boxed{\mathbb{1}_{(t\in(a,b))}}$

$T_r \sim \text{Gamma}(r, \lambda)$, $r, \lambda > 0$ $\quad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$

$T \sim \text{Exp}(\lambda)$, $\lambda > 0$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ $\leftarrow$ variable part

ex Name the distribution with the following variable part **ex** $\text{Gamma}(r = 1/2, \lambda = 3)$

a) $h(t) = t^3 e^{-\frac{1}{2}t}$ $\quad$ $\text{Gamma}(r = 4, \lambda = 1/2)$

b) $h(t) = e^{-\frac{1}{2}t^2}$ $\quad$ $\text{Normal}(\mu = 0, \sigma^2 = 1)$

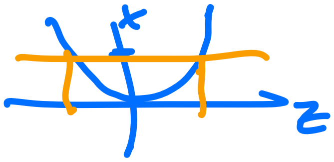
c) $h(t) = e^{-3t}$ $\quad$ $\text{Exp}(\lambda = 3)$

d) $h(t) = t^{-1/2} e^{-t}$ $\quad$ $\text{Gamma}(r = 1/2, \lambda = 1)$

e) $h(t) = 1_{(t \in (0,1))}$ $\quad$ $\text{Unif}(0,1)$

Stat 134

1. Let Z be a standard normal RV (with variable part $e^{-\frac{z^2}{2}}$). The variable part of the distribution $X = Z^2$ is?



[✓ **a**] Gamma $x^{-\frac{1}{2}}e^{-\frac{x}{2}}$

b Gamma $x^{\frac{1}{2}}e^{-\frac{x}{2}}$

c Exponential $e^{-\frac{x}{2}}$

d Normal $e^{-\frac{x^2}{2}}$

e none of the above

$$f_X(x) = \frac{f_Z(z)}{|g'(z)|}\Bigg|_{z=\sqrt{x}} + \frac{f_Z(z)}{|g'(z)|}\Bigg|_{z=-\sqrt{x}}$$

$$f_X(x) \underset{\uparrow}{\propto} \frac{e^{-\frac{x}{2}}}{2\sqrt{x}} + \frac{e^{-\frac{x}{2}}}{2\sqrt{x}} = \frac{1}{\sqrt{x}}e^{-\frac{x}{2}} = \boxed{x^{-1/2}e^{-x/2}}$$

Proportional to

(don't worry about constants)