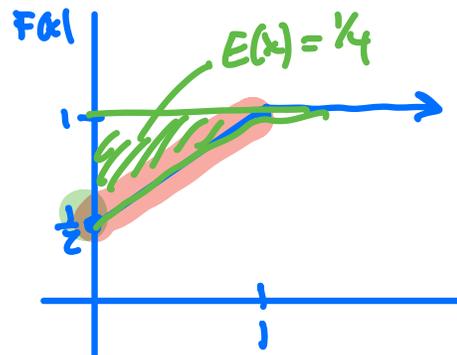
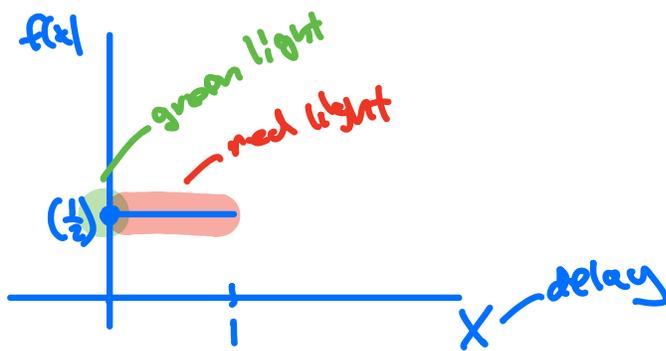


Wgetmap: 10:00-10:10

Suppose stop lights at an intersection alternately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let  $X$  be the delay of the car at the lights (assuming there is only one car on the road). Graph the density and the cdf of  $X$ . Also find  $E(X)$



Last time

sec 4.5 Expectation of a nonnegative RV using CDF

$$E(X) = \int_0^{\infty} (1 - F(x)) dx$$

ex let  $X \sim \text{Geom}(\frac{1}{2})$

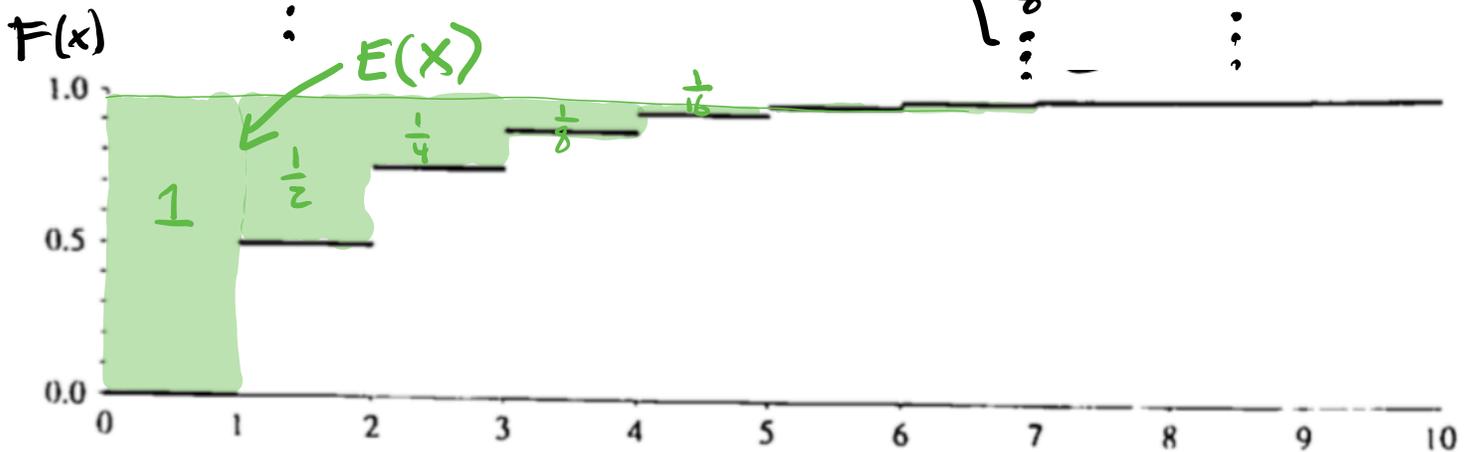
$$P(X=1) = \frac{1}{2}$$

$$P(X=2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$P(X=3) = (\frac{1}{2})^2 \cdot \frac{1}{2} = \frac{1}{8}$$

Picture

$$\Rightarrow F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ \frac{3}{4} & 2 \leq x < 3 \\ \frac{7}{8} & 3 \leq x < 4 \\ \vdots & \vdots \end{cases}$$



$$E(X) = \int_0^{\infty} (1 - F(x)) dx$$

$$E(X) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

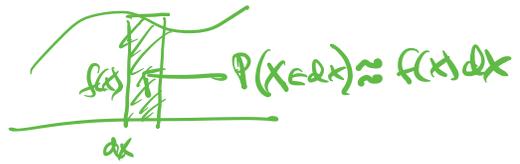
$= \sum_{j=0}^{\infty} P(X > j) = \sum_{j=0}^{\infty} (\frac{1}{2})^j \leftarrow \text{tail sum formula,}$

Today

- ① Overview of what we have learned since the midterm.
- ② sec 4.6 Order statistics

# ① Overview

## Chap 4



Single  
variable  
unconditional  
Prob

density of distribution  
change of variable formula for densities,  
expectation

continuous distributions

- uniform
- exponential/gamma
- order statistics / beta

MGF - useful tool  $\leftarrow$  calculate moments  
identity a distribution  
by its MGF

CDF / mixed distributions

calculating expectation from cdf.

## Chap 5

multiple  
variable  
unconditional  
Prob

joint distributions

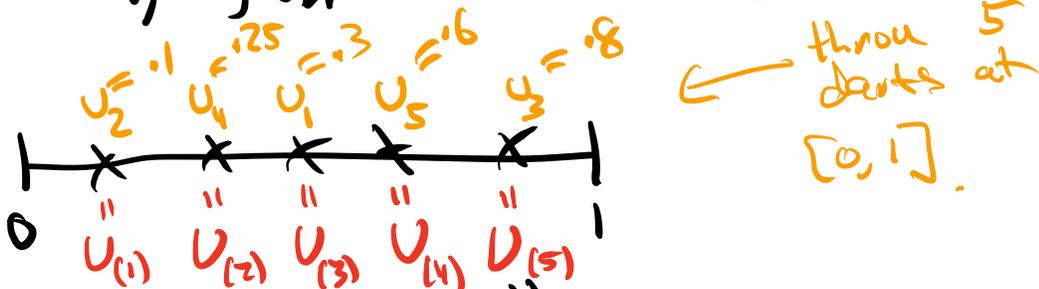
## Chap 6

multiple  
variable  
conditional  
Prob.

dependence

① Sec 4.6 order statistic of  $U(0,1)$

let  $U_1, \dots, U_n \stackrel{iid}{\sim} \text{Unit}(0,1)$



let  $U_{(k)}$  = called the  $k^{\text{th}}$  order statistic  
 =  $k^{\text{th}}$  largest value of  $U_1, \dots, U_n$   
 (assuming no ties)

or

$$U_{(1)} = \min(U_1, \dots, U_n)$$

$$U_{(n)} = \max(U_1, \dots, U_n)$$

Review counting

You have 3 red, 2 green and 5 blue marbles,  
 How many orderings of these 10 marbles are there?

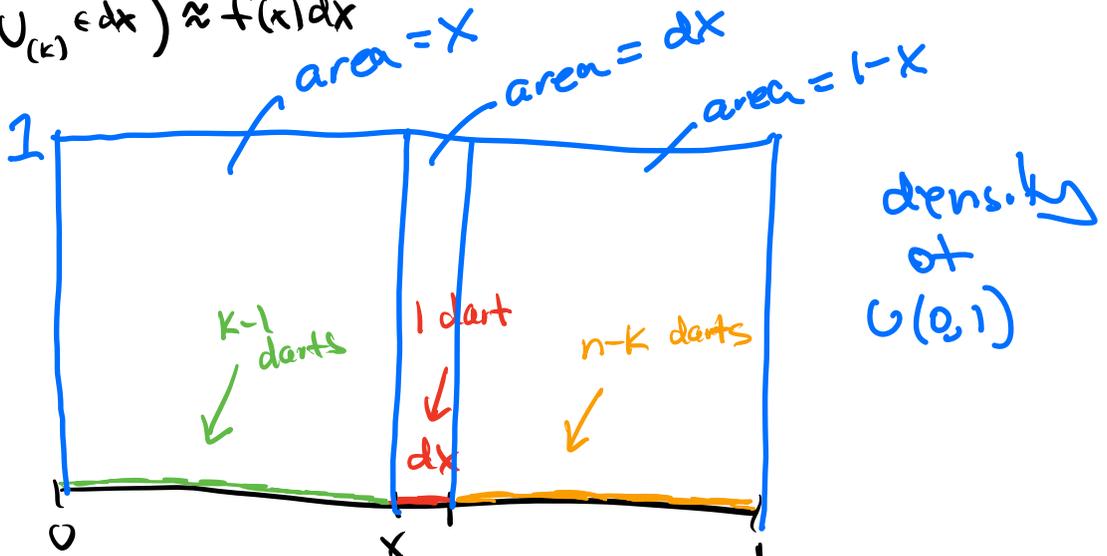
ex

rrr	gg	bbbbb	$\binom{10}{3,2,5} = \binom{10}{3} \cdot \binom{7}{2} \binom{5}{5}$ $\frac{10!}{3!2!5!}$
grrr	g	bbbbbb	
ggrrr	r	bbbbbb	
⋮	⋮	⋮	

Next, find density of  $U_{(k)}$

write  $P(U_{(k)} \in dx) \approx f(x) dx$

Picture



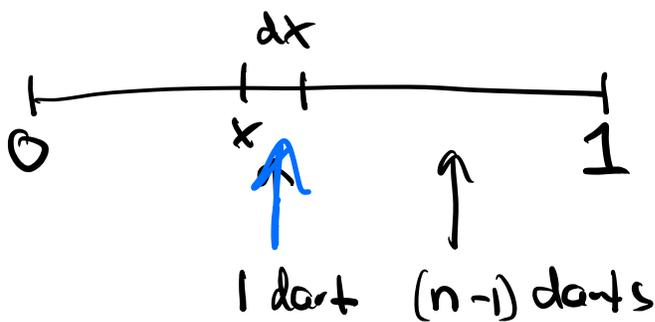
$U_{(k)} \in dx$  means that  $k-1$  darts are between 0 and  $x$ , and one is in  $dx$ , and  $n-k$  darts are between  $x$  and 1

$$\begin{aligned}
 P(U_{(k)} \in dx) &= P(k-1 \text{ darts} \in (0, x), 1 \text{ dart} \in dx, n-k \text{ darts} \in (x, 1)) \\
 &= P(k-1 \text{ darts} \in (0, x)) \cdot P(1 \text{ dart} \in dx \mid k-1 \text{ darts} \in (0, x)) \\
 &\quad \cdot P(n-k \text{ darts} \in (x, 1) \mid 1 \text{ dart} \in dx, k-1 \text{ darts} \in (0, x)) \\
 &= \binom{n}{k-1} x^{k-1} \binom{n-k+1}{1} dx \binom{n-k}{n-k} (1-x)^{n-k} \\
 &= \underbrace{\binom{n}{k-1, 1, n-k} x^{k-1} (1-x)^{(n-k+1)-1}}_{f_{U_{(k)}}(x)} dx
 \end{aligned}$$

$$\Rightarrow f_{U_{(k)}}(x) = \binom{n}{k-1, 1, n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{for } 0 < x < 1$$

or Let  $U_1, \dots, U_n \stackrel{iid}{\sim} \text{Unit}(0, 1)$

Find the density of  $U_{(1)}$

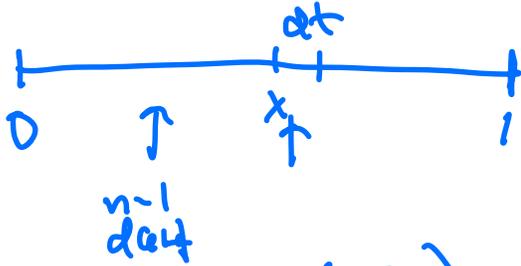


$$P(x_0 \leq dx) = \binom{n}{1, n-1} dx (1-x)^{n-1} = n(1-x)^{n-1} dx$$

$$\Rightarrow \boxed{f_{U_{(1)}}(x) = n(1-x)^{n-1}}$$

Ex Let  $U_1, \dots, U_n \stackrel{iid}{\sim} \text{Unif}(0,1)$

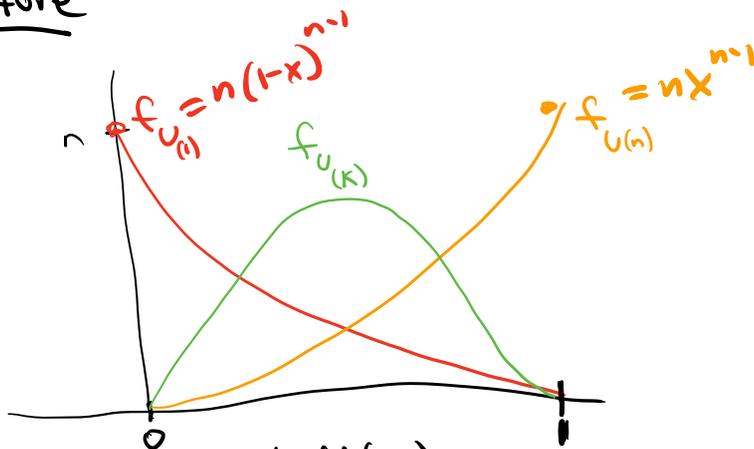
Find the density of  $U_{(n)}$



$$f_{U_{(n)}}(x) dx = \binom{n}{n-1, 1} x^{n-1} \cdot dx$$

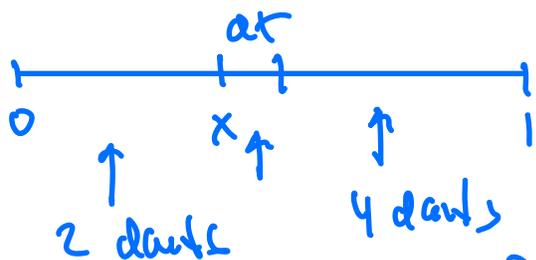
$$\Rightarrow f_{U_{(n)}}(x) = nx^{n-1}$$

Picture



Order statistic of  $U(0,1)$  provides a family of densities on the unit interval.

ex  $x^2(1-x)^4$  for  $0 < x < 1$  is the variable part of density of what RV? How many darts do you throw?



$U_{(3)}$  of  $n=7$

$$P(U_{(3)} \in dx) = f(x)dx \quad f_{U_{(3)}}(x) = \binom{7}{2,1,4} x^2 (1-x)^4$$

