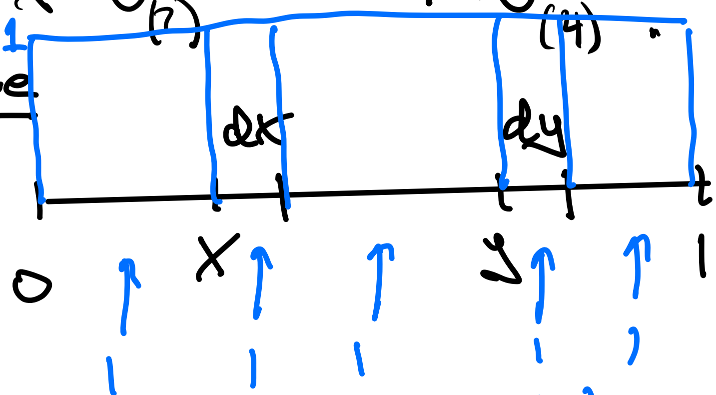
Stat 134 lec 27

Warmup 10:00–10:10

ex Throw down 5 darts on $(0,1)$.
Find the joint density, $f(x,y)$, of
$X = U_{(2)}$ and $Y = U_{(4)}$.

Picture



Hint:
Find $P(X \in dx, Y \in dy)$
$\approx f(x,y)\,dx\,dy$

$$P(X \in dx, Y \in dy) = \binom{5}{1} x \binom{4}{1} dx \binom{3}{1}(y-x)\binom{2}{1} dy \binom{1}{1}(1-y)$$

$$= \underbrace{\binom{5}{1,1,1,1,1} x(y-x)(1-y)}_{= f(x,y)}\,dx\,dy$$

$f(x,y)$ for $0 \le x, y \le 1$.

Last time

Sec 4.6 uniform order statistic

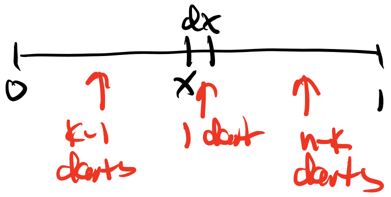
$U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$U_{(1)}, \dots, U_{(n)}$ order statistics

ex $U_1 = .2 \quad U_2 = .7 \quad U_3 = .1$

$\Rightarrow U_{(1)} = .1,\ U_{(2)} = .2,\ U_{(3)} = .7$

$P(U_{(k)} \in dx) = f(x)dx$



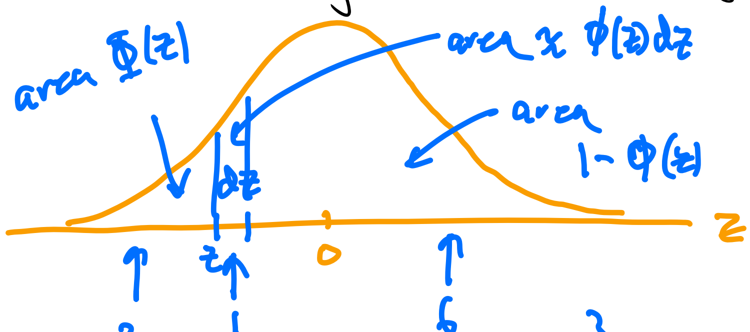
Note

$$\binom{n}{a,b,c} = \frac{n!}{a!\,b!\,c!} = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}$$

$$f_{U_{(k)}}(x) = \binom{n}{k-1,1,n-k} x^{k-1}(1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

normal order statistic

ex Let $Z_{(1)}, \dots, Z_{(10)}$ be the values of 10 independent standard normal variables arranged in increasing order. Find the density of $Z_{(4)}$



$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$\Phi(z) = P(Z \le z)$$

$$P(Z \in dz) = \binom{10}{3,1,6}\left(\Phi(z)\right)^3 \phi(z)dz \left(1-\Phi(z)\right)^6$$

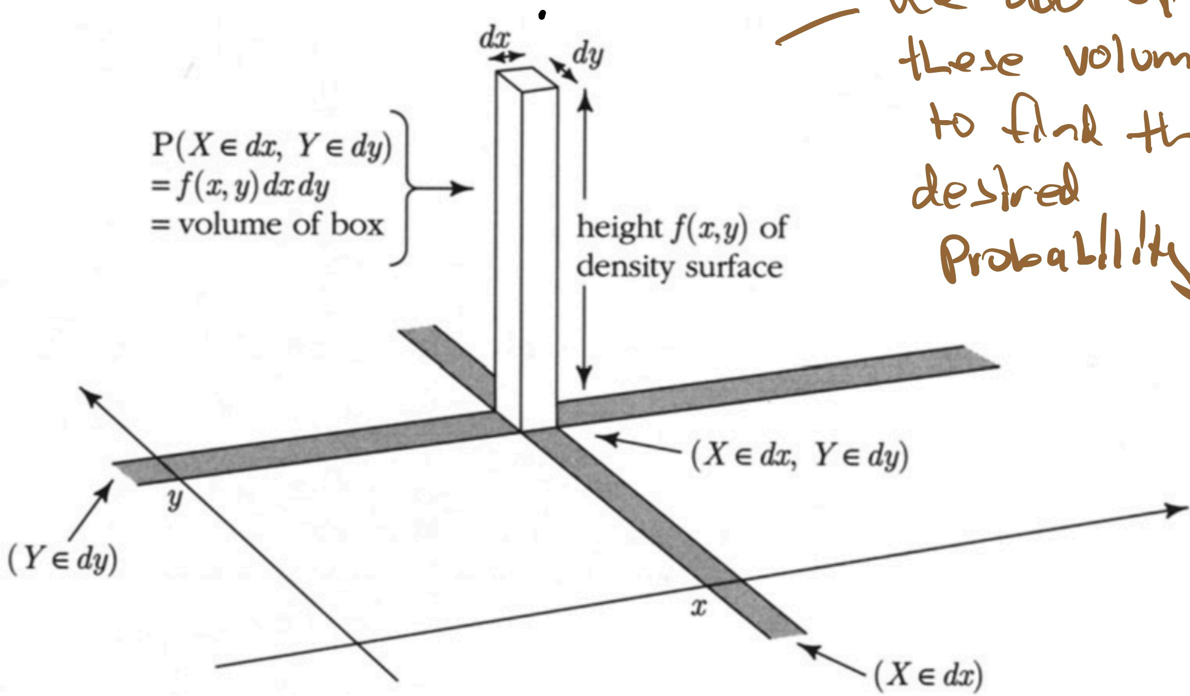
$\approx f(z)dz$

$$f(z) = \binom{10}{3,1,6}\left(\Phi(z)\right)^3 \phi(z)\left(1-\Phi(z)\right)^6$$

Today
1. Sec 5.1, 5.2 Continuous Joint Distribution
2. Sec 4.6 Beta distribution
3. Sec 5.1, 5.2 Calculate probabilities with $f(x,y)$.

① Sec 5.1, 5.2 Joint Density

$P(X \in dx, Y \in dy) \approx f(x,y)\,dx\,dy$



we add up these volumes to find the desired Probability.

$$\int_y \int_x f(x,y)\,dx\,dy = \int_x \int_y f(x,y)\,dy\,dx = 1$$

Stat 134

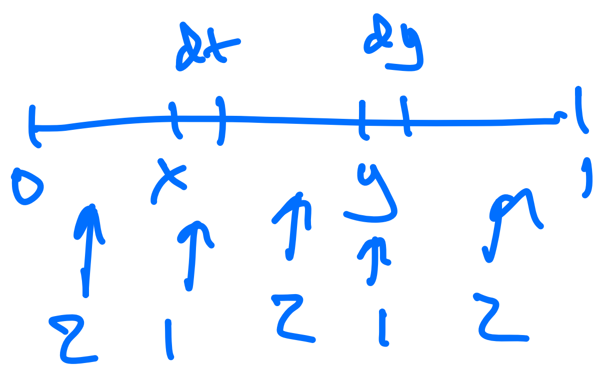
1. I throw down 8 darts on $(0,1)$. The variable part of the joint density of $X = U_{(3)}$ and $Y = U_{(6)}$ is:

   a $x(y-x)^5(1-y)^2$

   b $x^2(y-x)^2(1-y)^2$

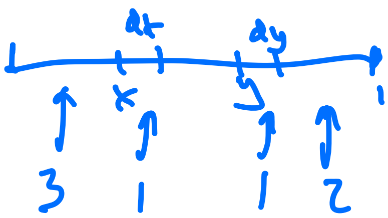
   c $x^4(y-x)^2(1-y)^2$

   d none of the above



Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1-y)^2$ for $0 < x < y < 1$.

Fill in the blanks: X and Y represent the __4__ smallest and __5__ smallest of __7__ i.i.d. Unif (0,1) random variables, respectively.



② Sec 4.6 Beta distribution.

$X \sim \text{Beta}(r,s)$ for $r>0$, $s>0$ is a distribution often used to model physical processes that take values between 0 and 1.

ex the proportion of defective items in a shipment.

Defn Let $r,s>0$. $X \sim \text{Beta}(r,s)$ if

$$f(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0 \leq x < 1.$$

where $\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$ Gamma function for $r>0$

or $\Gamma(r) = (r-1)!$ if $r \in \mathbb{Z}^+$

Notice if $r=1, s=1$, $f(x) = 1_{x \in (0,1)}$

$\Rightarrow \text{Beta}(1,1) = \text{Unif}(0,1)$.

ex Let $U_1, \dots, U_n \overset{iid}{\sim} U(0,1)$

$$f_{U_{(k)}}(x) = \binom{n}{k-1,1,n-k} x^{k-1}(1-x)^{(n-k+1)-1} \quad \text{on } 0<x<1$$

compare with,

$$f_{\text{Beta}(r,s)}(x) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} x^{r-1}(1-x)^{s-1} \quad \text{for } 0 \leq x < 1.$$

Notice that $f_{U_{(k)}}(x)$ and $f_{Beta(r,s)}(x)$ have the same variable part of their density when $r = k$

$s = n-k+1$

then $\Gamma(s+r) = \Gamma(n-k+1+k) = \Gamma(n+1) = n!$

$\Gamma(r) = (k-1)!$

$\Gamma(s) = (n-k)!$

$$\Rightarrow \frac{\Gamma(s+r)}{\Gamma(r)\Gamma(s)} = \binom{n}{k-1,\ 1,\ n-k}$$

$\Rightarrow$ Standard uniform ordered statistics are beta !

see appendix of notes

Thm $X \sim Beta(r,s)$

$$E(X) = \frac{r}{r+s}$$

Hence if $X \sim U_{(k)}$

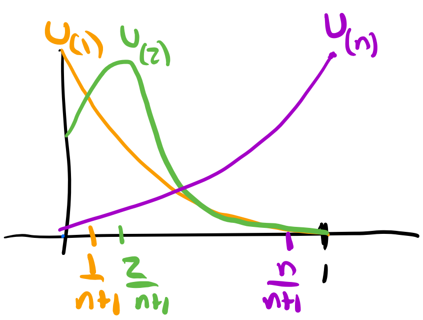
$$E(X) = \frac{k}{n-k+1+k} = \boxed{\frac{k}{n+1}}$$

$E(U_{(1)}) = \frac{1}{n+1}$

$E(U_{(2)}) = \frac{2}{n+1}$

$\vdots$

$E(U_{(n)}) = \frac{n}{n+1}$



ex (Bayesian Statistics)

Let P be the chance a coin lands head. Suppose the prior distribution of P is

$$f_P(p) = \begin{cases} c(1-p)^4 & \text{for } 0 \le p \le 1 \\ 0 & \text{else} \end{cases}$$

a) Is this a beta distribution? If so, what are the parameters?

$$f_{\text{Beta}(r,s)}(p) = \frac{\Gamma(s+r)}{\Gamma(s)\Gamma(r)} p^{r-1}(1-p)^{s-1} \text{ for } 0<p<1.$$

yes, compare $(1-p)^4$ and $p^{r-1}(1-p)^{s-1}$

$\Rightarrow P \sim \text{Beta}(1,5)$

b) Calculate the constant c

$$c = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} = \frac{\Gamma(6)}{\Gamma(1)\Gamma(5)} = \frac{5!}{0!\,4!} = 5$$

c) What is the mean of P?

$$E(P) = \frac{r}{r+s} = \frac{1}{1+5} = \boxed{1/6}$$

If $X \sim \text{Beta}(r,s)$

$$f(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}$$

Since $\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$

ex Let $X \sim \text{Beta}(3,4)$ $\left(\text{so } f(x) = \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2(1-x)^3\right)$

Compute $E(7X - 5X^6)$

$$E(7X-5X^6) = 7\underbrace{E(X)}_{= 3/7} - 5E(X^6)$$

$$E(X^6) = \int_0^1 x^6 \underbrace{\frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} x^2(1-x)^3}_{f(x)}\,dx$$

$$= \frac{\Gamma(7)}{\Gamma(3)\Gamma(4)} \int_0^1 x^8(1-x)^3\,dx \qquad \int_0^1 x^8(1-x)^3\,dx = \frac{\Gamma(9)\Gamma(4)}{\Gamma(13)}$$

$$= \frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}$$

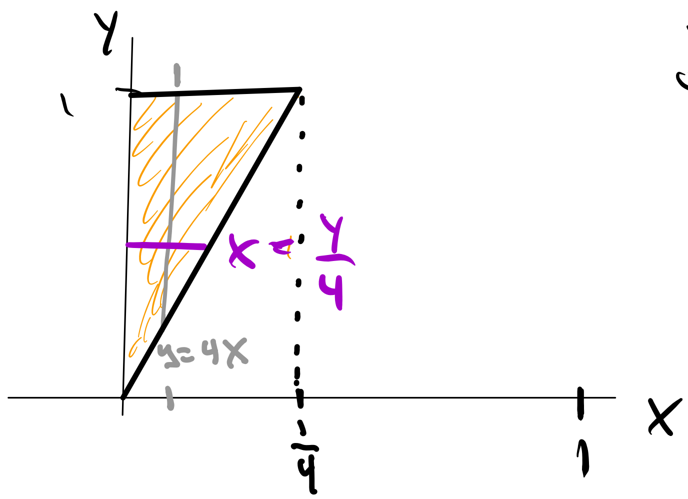
$$\Rightarrow E(7X-5X^6) = \boxed{3 - 5\frac{\Gamma(9)\Gamma(7)}{\Gamma(3)\Gamma(13)}}$$

③ Sec 5.1, 5.2 Calculate probabilities with $f(x,y)$.

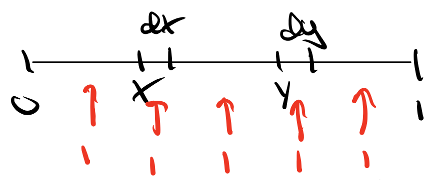
Throw down 5 darts on $(0,1)$.

ex $X = U_{(2)}$ $Y = U_{(4)}$

Find $P(Y > 4X)$



recall,



$$f(x,y) = \binom{5}{1,1,1,1,1} x\,(y-x)(1-y)$$

$$= \boxed{5!\, x\,(y-x)\,(1-y)}$$

$$P(Y>4X) = \int_{y=0}^{y=1} \int_{x=0}^{x=\frac{y}{4}} 5!\, x\,(y-x)\,(1-y)\,dx\,dy$$

(or)

$$P(Y>4X) = \int_{x=0}^{x=1/4} \int_{y=4x}^{y=1} 5!\, x\,(y-x)\,(1-y)\,dy\,dx$$

details:

$$P(Y>4x) = \int_{y=0}^{y=1}\int_{x=0}^{y/4} 120x(y-x)(1-y)\,dx\,dy$$

$$= \int_{y=0}^{y=1} 120(1-y)\int_{x=0}^{x=y/4}(xy-x^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1} 120(1-y)\left[\frac{x^2y}{2}-\frac{x^3}{3}\right]\Bigg|_{x=0}^{x=y/4} dy$$

$$= \int_{y=0}^{y=1} 120(1-y)\underbrace{\left(\frac{y^3}{32}-\frac{y^3}{3\cdot 64}\right)}_{\frac{6y^3-y^3}{3\cdot 64}=\frac{5y^3}{192}} dy$$

$$= \frac{5}{192}\cdot 120\int_{y=0}^{y=1} y^3-y^4\,dy$$

$$= \frac{5}{192}\cdot 120\int_0^1 y^3-y^4\,dy = \frac{5\cdot 120}{192}\left(\frac{y^4}{4}-\frac{y^5}{5}\right)\Bigg|_0^1$$

$$= \frac{5\cdot 120}{192}\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{30}{192} = .156$$

Appendix

Let $X \sim \text{Beta}(r,s)$

then $E(X) = \dfrac{r}{r+s}$.

Pf/ Note that $\displaystyle\int_0^1 f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = 1$

$$\Rightarrow \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

$$E(X) = \int_0^1 x f(x)\,dx = \frac{\Gamma(r+s)}{\Gamma(r)\cancel{\Gamma(s)}} \underbrace{\int_0^1 \overset{x^{r+1-1}}{x\, x^{r-1}}(1-x)^{s-1}\,dx}_{= \frac{\cancel{\Gamma(s)}\Gamma(r+1)}{\Gamma(s+r+1)}}$$

$$= \frac{(r+s-1)!\; r!}{(s+r)!} = \boxed{\frac{r}{r+s}}$$

$\square$