Stat 134 lec 29

warmup 10:00–10:10

ex (5.2.9a)

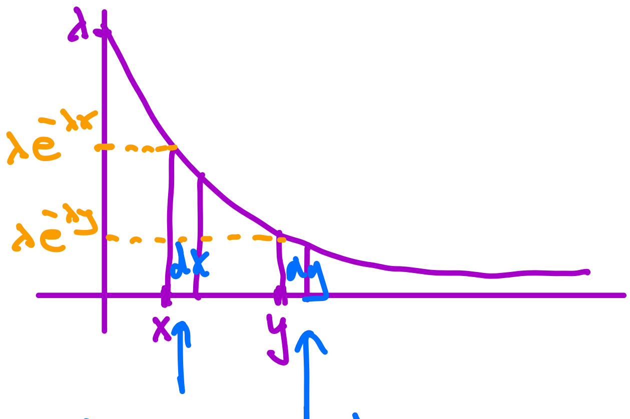
$S, T \overset{iid}{\sim} Exp(\lambda)$ $\left(f_S(s) = \lambda e^{-\lambda s} \right)$

$X = \min(S,T)$ ← 1st ordered statistic of Exp(λ)

$Y = \max(S,T)$ ← 2nd ordered statistic of Exp(λ)

Find the joint density of X and Y

Picture



$P(X \in dx, Y \in dy) \approx f(x,y)\,dx\,dy$

$$\binom{2}{1} \lambda e^{-\lambda x} dx \cdot \binom{1}{1} \lambda e^{-\lambda y} dy = 2\lambda^2 e^{-\lambda(x+y)} dx\,dy$$

$$\Rightarrow \boxed{f(x,y) = 2\lambda^2 e^{-\lambda(x+y)} \quad \text{for } 0 \le x \le y}$$

Earlier material

$T \sim Exp(\lambda)$, $cT \sim Exp\left(\frac{\lambda}{c}\right)$

Competing exponentials

$T_1 \sim Exp(\lambda_1)$, $T_2 \sim Exp(\lambda_2) \Rightarrow P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

Properties of std normal Z	Proved it
$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$	no
$E(Z) = 0$ — Even though Z is symmetric around zero it is possible $E(Z)$ is undefined. ex Cauchy distribution	no
$SD(Z) = 1$	no

Let $X, Y \overset{iid}{\sim} N(0,1)$ with density $\phi(x) = ce^{-\frac{1}{2}x^2}$, $c>0$

$f(x,y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ for $c>0$

we still need to show that $c = \frac{1}{\sqrt{2\pi}}$, $E(X)=0$, $SD(X)=1$.

Last time

sec 5.2 Marginal density $f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx$

ex

joint density

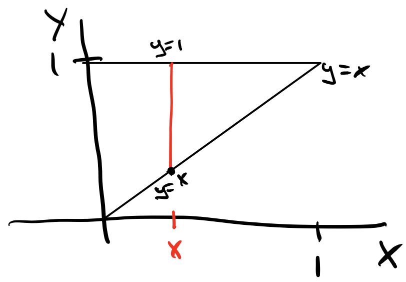
$$f(x,y) = \begin{cases} 30(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$

marginal density

$$f_X(x) = \int_{y=-\infty}^{y=\infty} f(x,y)\,dy$$



$$= \int_{y=x}^{y=1} 30(y-x)^4\,dy$$

$u = y-x$
$du = dy$

$$= \int_{u=0}^{u=1-x} 30u^4\,du = \frac{30u^5}{5}\Big|_0^{1-x} = \boxed{6(1-x)^5,\ 0<x<1}$$

* In appendix we show $f_Y(y) = \boxed{6y^5}$ $0<y<1$

$$f(x,y) = 30(y-x)^4 \neq \underset{\underset{6(1-x)^5}{\parallel}}{f_X(x)}\ \underset{\underset{6y^5}{\parallel}}{f_Y(y)}$$

so $X = Y_{(1)}$ and $Y = U_{(6)}$ are dependent,

Today
1. Sec 5.2 Marginal Densities
1. Sec 5.2 Expectation $E(g(X,Y))$
2. Sec 5.3 Rayleigh distribution

① Sec 5.2 Marginal Densities

Stat 134

Friday November 8 2019

1. S and T are i.i.d. Exp(λ). $X = \text{Min}(S, T)$ and $Y = \text{Max}(S, T)$. The joint density is $f(x, y) = 2\lambda^2 e^{-\lambda(x+y)}$. The marginal density of Y is:

   a $\lambda(1 - e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   b $2\lambda(1 - e^{-\lambda y})e^{-\lambda y}$ for $y > 0$

   c $2\lambda(1 - e^{-\lambda y})$ for $y > 0$

   d none of the above

$$f_Y(y) = 2\lambda^2 e^{-\lambda y} \underbrace{\int_{x=0}^{x=y} e^{-\lambda x}\,dx}_{\left.\frac{e^{-\lambda x}}{-\lambda}\right|_0^y = \frac{1-e^{-\lambda y}}{\lambda}} = \boxed{2\lambda(1-e^{-\lambda y})e^{-\lambda y} \text{ for } y>0}$$

② Sec 5.2 Expectation $E(g(X,Y))$

Let (X,Y) have joint density $f(x,y)$ and $g(X,Y)$ be a function of X,Y.

Define

$$E(g(X,Y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\, dx\, dy$$

ex

joint density
$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$
$Y = U_{(6)}$

Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\, dx\, dy$$

$g(X,Y) = Y$

we know $Y = U_{(6)} \sim \text{Beta}(6,1) \Rightarrow E(Y) = \frac{6}{6+1} = \boxed{\frac{6}{7}}$

k $\quad n-k+1 = 6-6+1 = 1$

See appendix to notes.

③ Sec 5.3 Rayleigh Distribution

let $T \sim \text{Exp}(\frac{1}{2})$, $f(t) = \frac{1}{2} e^{-\frac{1}{2}t}$, $t > 0$

$R = \sqrt{T}$ ← R is called the Rayleigh Distribution

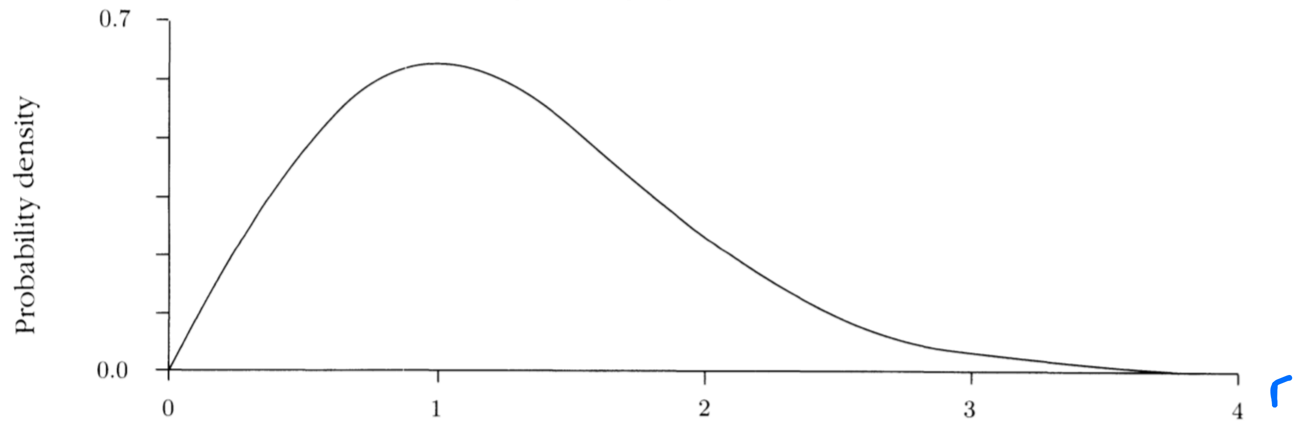
write $R \sim \text{Ray}$

Find $f_R(r)$.

$$f_R(r) = \left.\frac{f_T(t)}{\left|(\sqrt{t})'\right|}\right|_{t=r^2}$$

$$= \frac{\frac{1}{2} e^{-\frac{1}{2}r^2}}{\frac{1}{2r}} = \boxed{r e^{-\frac{1}{2}r^2}, \ r > 0}$$

FIGURE 3. Density of the Rayleigh distribution of R.



Note:

$$P(R > r) = P(R^2 > r^2) = P(\overset{\sim \text{Exp}(\frac{1}{2})}{T} > r^2)$$

$$= e^{-\frac{1}{2}r^2}$$

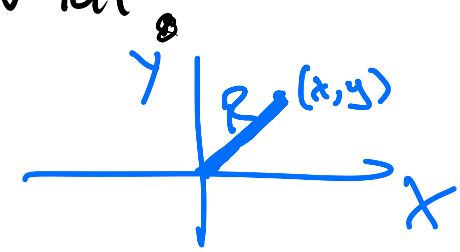
So $\boxed{F_R(r) = 1 - e^{-\frac{1}{2}r^2}, \quad r \geq 0}$

$$f_R(r) = \frac{d}{dr} F_R(r)$$

$$= 0 + \frac{1}{2} e^{-\frac{1}{2}r^2} \cdot 2r = \boxed{re^{-\frac{1}{2}r^2}, \quad r \geq 0}$$

The Rayleigh distribution will help us find the density of the standard normal.

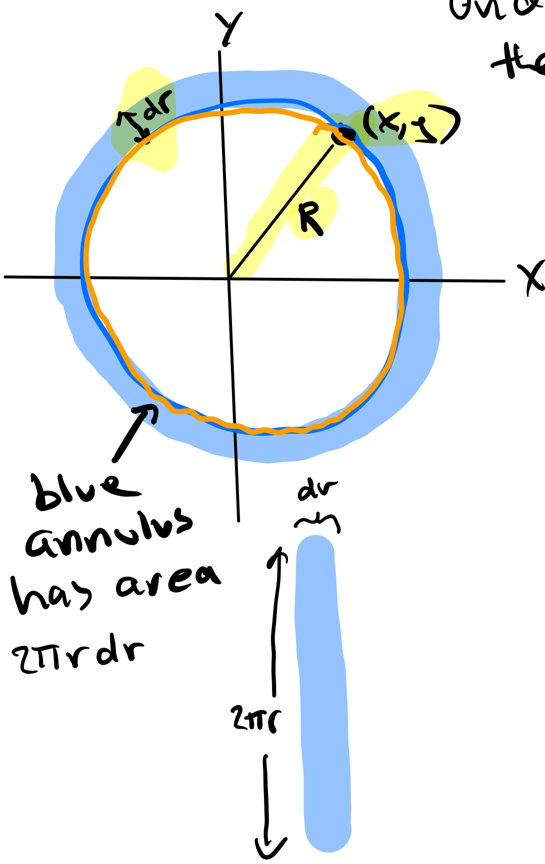
For $X, Y \overset{iid}{\sim} N(0,1)$



Let $R = \sqrt{X^2 + Y^2}$

We will show that $R \sim \text{Ray}$

$P(R \in dr)$ is the volume of the cylinder under $f(x,y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ over the shaded blue annulus.



blue annulus has area $2\pi r\, dr$

$$P(R \in dr) \approx \text{height of } f(x,y) \text{ above blue annulus}$$

$$= f_R(r)\,dr$$

$\cdot$ area of blue annulus

$$= c^2 e^{-\frac{1}{2}r^2} \cdot (2\pi r\, dr)$$

$$= c^2 2\pi\, r e^{-\frac{1}{2}r^2}\, dr$$

Rayleigh Density

$$1 = \int_{r=0}^{r=\infty} P(R \in dr) = c^2 2\pi \underbrace{\int_0^\infty r e^{-\frac{1}{2}r^2}\, dr}_{=1}$$

$$\Rightarrow c^2 \cdot 2\pi = 1$$

$$\Rightarrow \boxed{c = \frac{1}{\sqrt{2\pi}}}$$

Conclusions

1. For $X, Y \overset{iid}{\sim} N(0,1)$ and $T \sim \text{Exp}(\frac{1}{2})$
$R = \sqrt{X^2+Y^2}$ and $R = \sqrt{T}$
are both the Rayleigh distribution

little phi

2. $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is density of the standard normal.
(i.e. $c = \frac{1}{\sqrt{2\pi}}$)

- see end of lecture notes

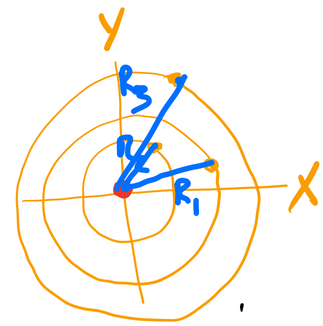
3. $E(X) = 0$ and $SD(X) = 1$

$\underline{\underline{\text{ex}}}$ Suppose 3 shots are fired at a target. Assume for each shot, both X and Y are standard normals. Let W be the closest distance among 3 shots to the bullseye. Find $f_W(w)$

distance to Bullseye

$$R = \sqrt{X^2+Y^2}$$

Picture



$$W = \min(R_1, R_2, R_3)$$

where $R_1, R_2, R_3 \overset{iid}{\sim}$ Ray

$$F_{R_1}(r) = 1 - e^{-\frac{1}{2}r^2}$$

$$P(W>w) = P(R_1>w,\ R_2>w,\ R_3>w)$$
$$= \left(P(R_1>w)\right)^3 = \left(e^{-\frac{1}{2}w^2}\right)^3 = e^{-\frac{3}{2}w^2}$$

$$F(w) = 1 - e^{-\frac{3}{2}w^2}$$

$$\boxed{f(w) = \frac{d}{dw}F(w) = 3we^{-\frac{3}{2}w^2},\ w \geq 0}$$

You can also do this using Rayleigh ordered statistics

$$P(W \in dw) = f(w)\,dw$$

$$= \binom{3}{1}\underbrace{w e^{-\frac{1}{2}w^2}}\,dw \cdot \binom{2}{2}\left(e^{-\frac{1}{2}w^2}\right)^2 = 3we^{-\frac{3}{2}w^2},\ w>0$$

Choice of 1 of the 3 Rayleighs

density of Rayleigh

survival function of two independent Rayleigh.

Appendix

ex

joint density

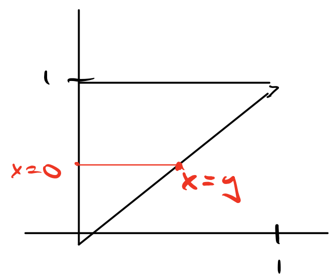
$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$

marginal density

Find $f_Y(y)$



$$= \int_{x=0}^{x=y} 30\,(y-x)^4\,dx$$

$u = y-x$

$du = -dx$

$$= -\int_{u=y}^{u=0} 30u^4\,du = \left.\frac{30u^5}{5}\right|_0^y = \boxed{6y^5} \quad 0<y<1$$

$f(x,y) = 30(y-x)^4 \neq f_X(x)\,f_Y(y)$

$f_X(x) = 6(1-x)^5$, $f_Y(y) = 6y^5$

so $X = U_{(1)}$ and $Y = U_{(6)}$ are dependent, Appendix

Expectation $E(g(X,Y))$

ex

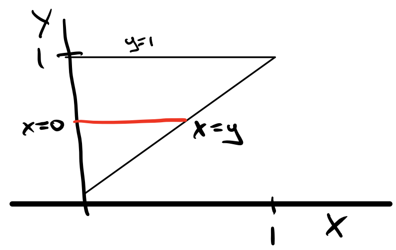
$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\,dx\,dy$$

joint density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$



Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\,dx\,dy$$

$g(x,y) = Y$

$$= \int_{y=0}^{y=1} y \int_{x=-\infty}^{x=\infty} f(x,y)\,dx\,dy$$

$f_Y(y) = 6y^5$

showed this in lecture 29

$$= 6\int_{y=0}^{y=1} y^6\,dy = \frac{6y^7}{7}\Big|_0^1 = \boxed{6/7}$$

Appendix

Claim Let $X \sim N(0,1)$. We show that $E(X)=0$ and $SD(X)=1$.

Let $X \sim N(0,1)$

Show $E(X)=0$

$\phi(x)$ is symmetric about zero so all we have to show is that $E(X)$ converges absolutely.

(i.e $E(|X|) < \infty$).

$E(X) = \int x \underbrace{f(x)}_{\phi(x)} dx$

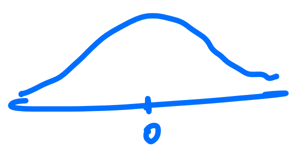
$$E(|X|) = \int_{-\infty}^{\infty} |x| \phi(x) dx$$

$$= 2 \int_0^{\infty} x \phi(x) dx$$

$$= 2 \int_0^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \underbrace{\int_0^{\infty} x e^{-\frac{x^2}{2}} dx}_{= 1}$$

Rayleigh density



$E(|X|) = 2\int_0^{\infty} x f(x) dx < \infty$ ✓

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{0} x f(x) dx + \int_0^{\infty} x f(x) dx$$

∞ ∞

$$= \sqrt{\frac{2}{\pi}} < \infty.$$

$\Rightarrow E(X) = 0$ since $\phi(x)$ is semetric around zero

Next we show $SD(X) = 1$:

We know $X^2 + Y^2 \sim Exp\left(\frac{1}{2}\right)$ from above
λ

so $E(X^2+Y^2) = \dfrac{1}{\frac{1}{2}} = 2$

λ rate of X^2+Y^2

$\Rightarrow E(X^2) + \underset{E(X^2)}{E(Y^2)} = 2$

$\Rightarrow 2E(X^2) = 2 \Rightarrow E(X^2) = 1$

but $SD(X) = \sqrt{E(X^2) - E(X)^2}$

$= \sqrt{1 - 0} = 1$ ✓