Stat 134 lec 31

Warmup 10:00 – 10:10

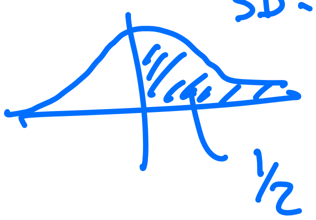
Let $X, Y \overset{iid}{\sim} N(0,1)$

Find $P(X > 2Y)$

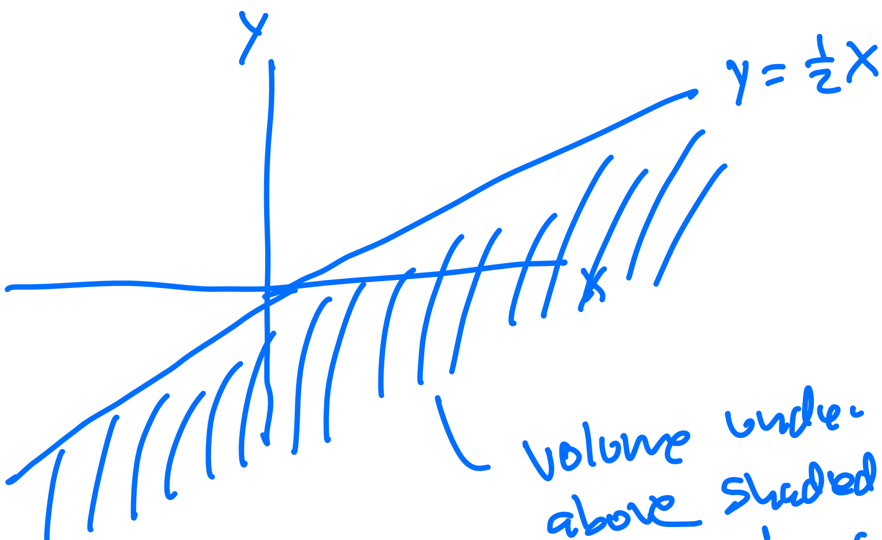
$= P(X - 2Y > 0)$

$X - 2Y \sim N(0, 5)$

$SD = \sqrt{5}$



$P(X - 2Y > 0) = 1/2$



volume under joint $f(x,y)$ above shaded region is 1/2 by symmetry of bell shaped density.

Last time

Sec 5.3

A linear combination of independent normals is normal.

Thm Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ $\Big\}$ indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2,\ a^2\sigma_1^2 + b^2\sigma_2^2)$

Note In Chapter 6 we will generalize this result and show that $aX_1 + bX_2$ is normal iff (X_1, X_2) are bivariate normal

Sec 5.4 Convolution formula for density of sum

ex Let X and Y be discrete RVs

$$P(X+Y=z) = \sum_{\text{all } x \in X} P(X=x, Y=z-x)$$

ex $X, Y \overset{iid}{\sim} \text{Geom}(1/4)$ on $1, 2, 3, \ldots$

$X \sim \text{Geom}(p)$ on $1, 2, \ldots$

$P(X=n) = q^{n-1} p$

$$P(X+Y=4) = P(1,3) + P(2,2) + P(3,1)$$

$$= P(1)P(3) + P(2)P(2) + P(3)P(1)$$

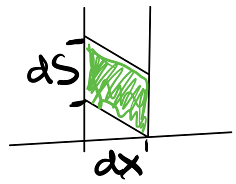
$$= \frac{1}{4}\left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} \qquad \left(\frac{3}{4}\cdot\frac{1}{4}\right)^2 \qquad \left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)\cdot\frac{1}{4}$$

Sec 5.4

1. Convolution formula for the density of $X+Y$
2. triangular density
3. Uniform spacing (see #13 p 355)

(1) Sec 5.4 The Density Convolution Formula

A little geometry:



Area of parallelogram
$A = dx\,dS$

Let $X \geq 0$, $Y \geq 0$ be continuous RVs with joint density $f(x,y)$.

Let $S = X + Y$

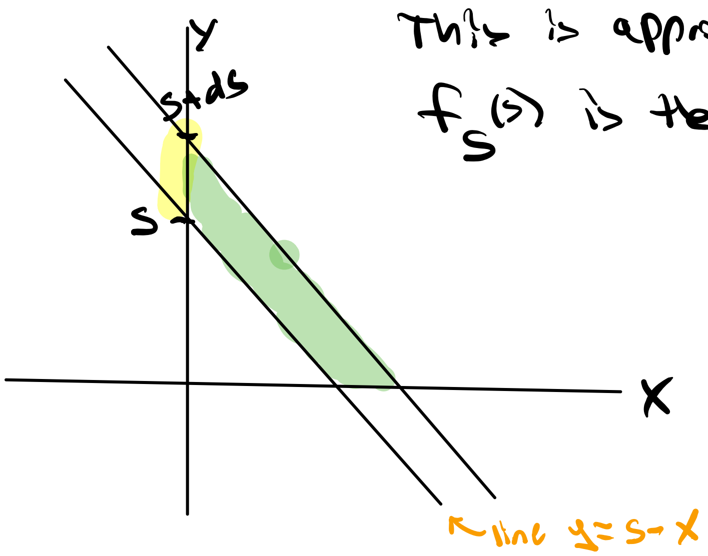
Find the density of S

$s = x + y$
$y = s - x$

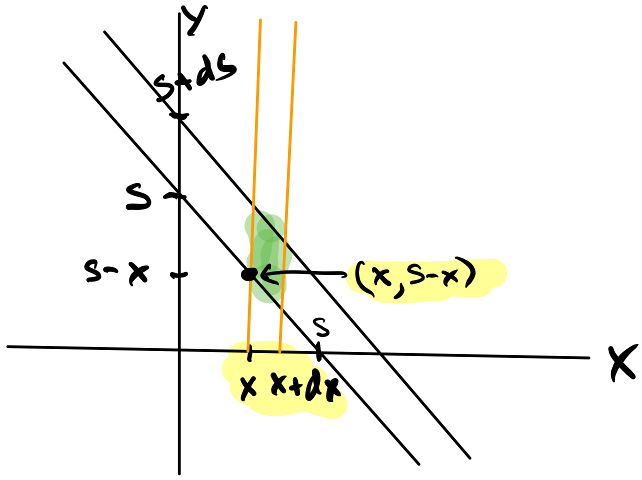
y intercept

$P(S \in ds)$ is the volume under $f(x,y)$ over the green region.

This is approx $f_S(s)\,ds$ where $f_S(s)$ is the density of S.



$P(X \in dX, S \in dS)$ is the volume under $f(x,y)$ over the green region.



$$\underset{\substack{\approx \\ f_S(s)\,ds}}{P(S \in ds)} = \int_{x=0}^{x=s} \overset{\substack{f(x,s-x)\,dx\,ds \\ \approx}}{P(X \in dx, S \in dS)}$$

$$= \underbrace{\int_{x=0}^{x=s} f(x, s-x)\,dx}_{f_S(s)}\,ds$$

$$\Rightarrow f_S(s) = \int_{x=0}^{x=s} f(x, s-x)\,dx$$

convolution formula for densities.

Compare with:

$$P(S=s) = \sum_{x=0}^{s} P(x, s-x)$$

convolution formula for p.m.f

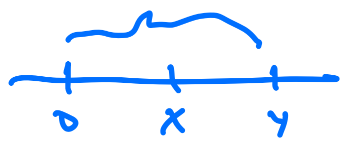
Ex $X, Y \overset{iid}{\sim} \text{expon}(\lambda)$ $S = X+Y$

$X \sim \exp(\lambda)$

$f_X(x) = \lambda e^{-\lambda x}$

$$f_S(s) = \int_0^s f(x, s-x)\,dx$$

fixed



$$= \int_0^s f_X(x)\, f_Y(s-x)\,dx$$

$$= \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}\,dx$$

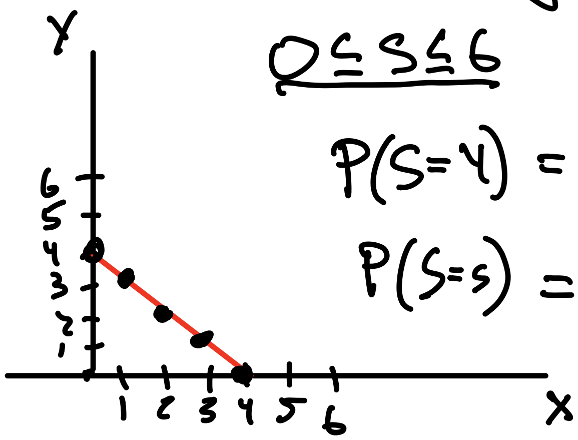
$$= \lambda^2 e^{-\lambda s} \int_0^s dx = \lambda^2 e^{-\lambda s} \cdot x \Big|_0^s = \boxed{\lambda^2 s e^{-\lambda s}}$$

$S \sim \text{Gama}(2, \lambda)$ ✓

(2) Sec 5.4 Triangular density

Let $\left.\begin{array}{l} X \sim \text{Unif}\{0,1,2,\dots,6\} \\ Y \sim \text{Unif}\{0,1,2,\dots,6\} \end{array}\right\}$ indep.

Find probability mass function of $S = X+Y$



$\underline{0 \le S \le 6}$

$$P(S=4) = \underbrace{P(0,4)}_{=\left(\frac{1}{7}\right)^2} + \underbrace{P(1,3)}_{=\left(\frac{1}{7}\right)^2} + P(2,2) + P(3,1) + P(4,0)$$

$= 5/49$

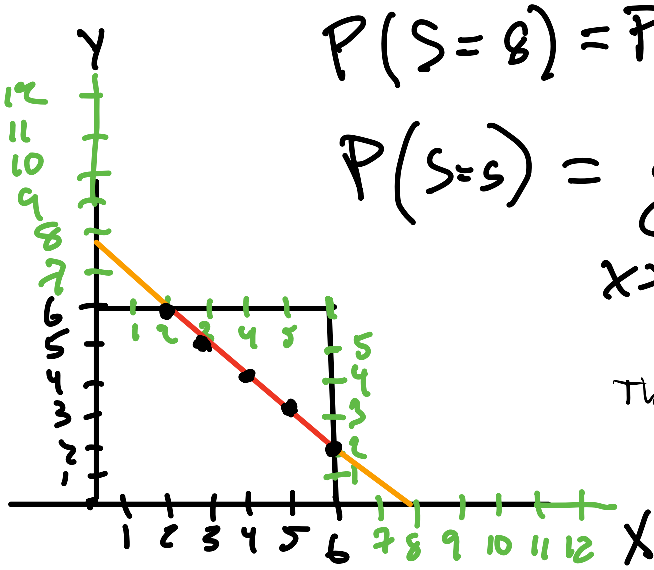
$$P(S=s) = \sum_{x=0}^{s} P(X=x, Y=s-x)$$

$\underline{7 \le S \le 12}$

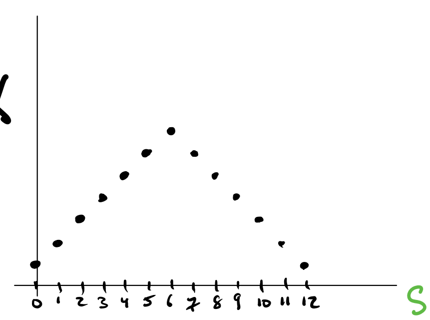
$$P(S=8) = P(2,6) + P(3,5) + \dots + P(6,2)$$

$= 5/49$

$$P(S=s) = \sum_{x=s-6}^{6} P(x, s-x)$$



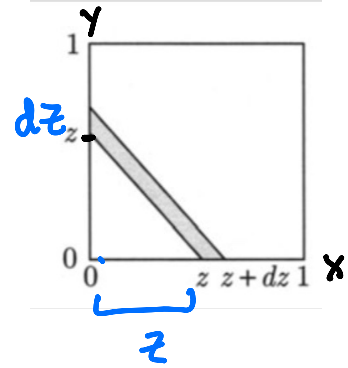
The distribution of $S = X+Y$ looks like



Continuous case:

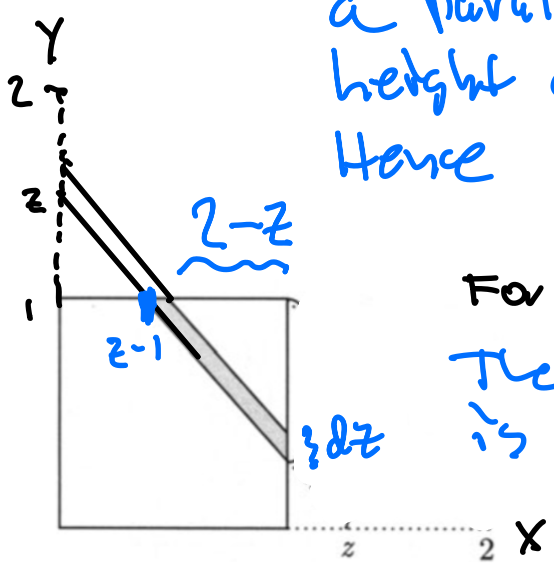
$X \sim U(0,1)$, $Y \sim U(0,1)$ } indep

Find density of $Z = X + Y$



For $0 < z < 1$

$P(Z \in dz)$ is the volume under $f(x,y) = 1$ above the shaded region. The shaded region is approximately a parallelogram w/ width z and height dz so area $\approx z\,dz$. Hence $P(Z \in dz) = 1 \cdot z\,dz \Rightarrow f_Z(z) = z$ for $0 < z < 1$



For $1 < z < 2$

The area of the shaded region is $(2-z)\,dz$ so

$P(Z \in dz) = 1\,(2-z)\,dz$

$\Rightarrow f_Z(z) = 2 - z$ for $1 < z < 2$