Stat 134 lec 33

Warmup 10:00–10:10

8 transistors (type 1) are distributed $Exp(\frac{1}{100})$ and 4 transistors (type 2) are $Exp(\frac{1}{200})$.
Let T be the lifetime of a randomly picked transistor.

a) Find $E(T \mid \text{transister is type } 1)$

$T \mid \text{transister type } 1 \sim Exp(1/100)$

$E(T \mid \text{transister } 1) = 100$ hrs

b) Find $E(T)$

X = type of transistor

$E(T) = E(E(T \mid X))$

$E(T) = E(T \mid X=1)P(X=1) + E(T \mid X=2)P(X=2)$

$= 100 \cdot \frac{8}{12} + 200 \cdot \frac{4}{12} = \boxed{133.3}$

Last time

Sec 5.4 Uniform Spacing

You ran only throw n dart at $[0,1]$.

For $0 < a < a+k \le n$, $U_{(a+k)} - U_{(a)} = U_{(k)} =$ Beta $(k, n-k+1)$

Sec 5.4 Convolution formula for density of ratio Y/X

$X > 0$, $Y > 0$

let $Z = \frac{Y}{X}$.

Convolution formula.

$$f_Z(z) = \int_{x=0}^{x=\infty} f(x, zx)\, x \, dx = \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\, x \, dx$$

- if X, Y indep.

Sec 6.1

Rule of average conditional probabilities (discrete case)

Let X and N be discrete RV w/ joint distribution $P(X=x, N=n)$.

$$P(X=x) = \sum_n P(X=x, N=n)$$

$$= \sum_n P(X=x \mid N=n) P(N=n)$$

Today

1. Sec 5.4 General convolution formula
2. Sec 6.2 Properties of conditional expectation.
3. Sec 5.4 Uniform Spacing continued

① sec 5.4 General convolution formula

We have different convolution formulas for sums and quotients.

We can write a general convolution formula for any operation.

1 dimensional change of variables

RV	transformed RV
Y	$z(Y)$ ← a differentiable function
f_Y	$f_Z = \left\lvert \frac{\partial y}{\partial z} \right\rvert f_Y$

$\frac{1}{\left\lvert \frac{\partial z}{\partial y} \right\rvert}$

ex $Z = Y^3$

2 dimensional change of variables

RV	transformed RV
(X, Y)	$(X, Z(X,Y))$ ← a differentiable function
$f_{X,Y}$	$f_{X,Z} = \left\lvert \det \frac{\partial(X,Y)}{\partial(X,Z)} \right\rvert f_{X,Y}$

ex $Z = X+Y$

ex $Z = \frac{X}{Y}$, $Y \neq 0$

$$= \left\lvert \det \begin{bmatrix} \underbrace{\frac{\partial x}{\partial x}}_{=1} & \frac{\partial x}{\partial z} \\ \underbrace{\frac{\partial y}{\partial x}}_{=0} & \frac{\partial y}{\partial z} \end{bmatrix} \right\rvert f_{X,Y}$$

$$= f_{X,Y} \left\lvert \frac{\partial y}{\partial z} \right\rvert$$

Convolution formula

let $Z(X,Y)$ be a differentiable function of X, Y

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Z}(x,z)\, dx = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left\lvert \frac{\partial y}{\partial z} \right\rvert dx$$

ex Let $Z = \frac{Y}{X}$. Find the convolution formula for Z.

$\Rightarrow Y = XZ \Rightarrow \frac{\partial y}{\partial z} = x$

$$\Rightarrow f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, xz)\,|x|\,dx$$

Convolution formula for quotient.

ex Let $Z = \frac{X}{X+Y}$. Find the convolution formula for Z.

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\left|\frac{\partial y}{\partial z}\right| dx$$

Step 1 Solve for Y

$$Zx + Zy = x \Rightarrow Zy = x - Zx$$

$$\Rightarrow y = \frac{x(1-z)}{z}$$

Step 2 Find $\frac{\partial y}{\partial z}$

$$\frac{\partial y}{\partial z} = x\left[\frac{(1-z)'z - (1-z)z'}{z^2}\right] = x\left[\frac{-z-1+z}{z^2}\right] = \boxed{\frac{-x}{z^2}}$$

Step 3 Substitute $y, \frac{\partial y}{\partial z}$ in $f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\left|\frac{\partial y}{\partial z}\right| dx$

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(x, \frac{x(1-z)}{z}\right)\frac{|x|}{z^2}\,dx}$$

Sec 6.2 Conditional Expectation (discrete case)

Bayes rule:

recall $P(T=t \mid S=s) = \dfrac{P(T=t, S=s)}{P(S=s)}$

ex (T,S) is joint distribution below,

Find $P(T=3 \mid S=7)$

$$\frac{P(T=3, S=7)}{P(S=7)} = \frac{.3}{.4} = \boxed{.75}$$

	T=3	T=4	Sum ← marginal of S
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

↑ marginal of T

Find $P(T=4 \mid S=7)$

$$\frac{P(T=4, S=7)}{P(S=7)} = \frac{.1}{.4} = \boxed{.25}$$

Find $E(T \mid S=7)$

$$\sum_{t \in T} t\,P(T=t \mid S=7) = 3 \cdot P(T=3 \mid S=7) + 4 \cdot P(T=4 \mid S=7)$$

$$= 3(.75) + 4(.25) = \boxed{3.25}$$

Find $E(T \mid S=6)$

$3 \cdot P(T=3 \mid S=6) + 4 \cdot P(T=4 \mid S=6)$

$= 3\left(\frac{.2}{.4}\right) + 4\left(\frac{.2}{.4}\right) = 3.5$

	T=3	T=4	Sum ← marginal of S
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

marginal of T

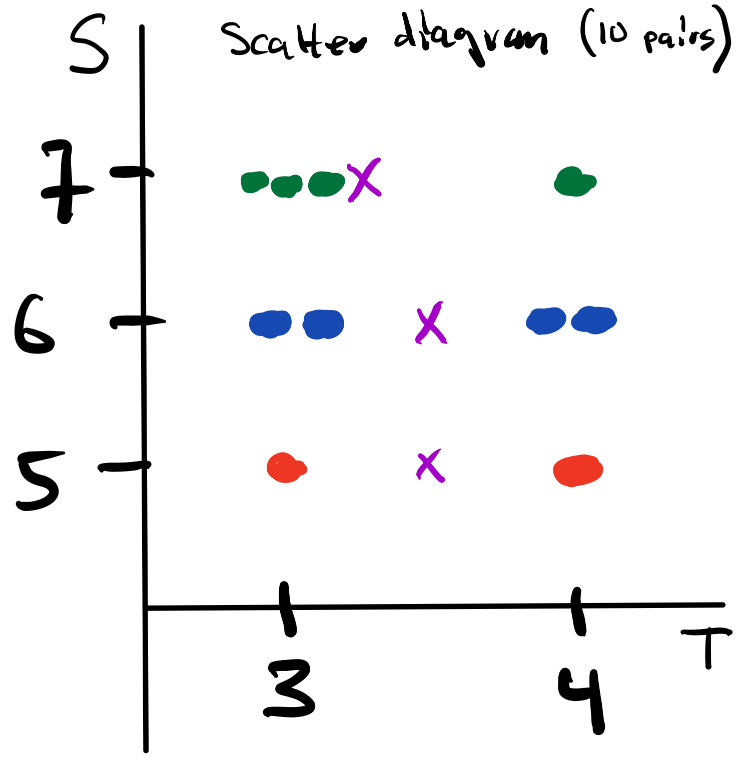
$$\left.\begin{aligned} E(T \mid S=7) &= 3.25 \\ E(T \mid S=6) &= 3.5 \\ E(T \mid S=5) &= 3.5 \end{aligned}\right\} \text{function of } S$$

Picture

joint distribution

	T=3	T=4	sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
sum	0.6	0.4	1

$\longleftrightarrow$



Two main points:

① $E(T|S)$ is a function of S.

② $E(T|S)$ is a RV so it has an expectation.

Next we explore the expectation of $E(T|S)$.

Let $g(S) = E(T|S)$

$g(7) = E(T|S=7) = 3.25$

$g(6) = 3.5$

$g(5) = 3.5$

$$E(g(S)) = \sum_{s \in S} g(s) P(S=s)$$

$$= 3.25(.4) + 3.5(.4) + 3.5(.2)$$

$$= 3.4$$

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1

Find $E(T)$

$$E(T) = \sum_{t \in T} t P(T=t) = 3(.6) + 4(.4) = 3.4$$

In other words,

$$E(E(T|S)) = E(T)$$

This is called the property of <u>iterated expectations</u>.

Intuitively,

If you have a class that is $2/3$ girls and $1/3$ boys and the girls weigh on average 100 lbs and boys weigh 200 lbs then the average weight of the class should be $\frac{2}{3}(100) + \frac{1}{3}(200)$. i.e. we take the weighted average of the averages.

Rule of average conditional expectations

For any random variable T with finite expectation and any discrete RV S,

$$E(T) = \sum_{\text{all } s} E(T \mid S=s) \cdot P(S=s)$$

(see end of this lecture for a formal proof)

③ **Uniform spacing continued**

let $U_1, \ldots, U_{10} \overset{iid}{\sim} \text{Unif}(0,1)$

and $U_{(1)}, \ldots, U_{(10)}$ be ordered standard uniform.

let $Y \sim U_{(3)}$, $X \sim U_{(5)}$

Let $Z = \frac{Y}{X}$

What distribution is Z?

ex

Suppose $U_{(1)} = .1$, $U_{(2)} = .2$, $U_{(3)} = .3$, $U_{(4)} = .4$, $U_{(5)} = .5$

then

$$\frac{U_{(1)}}{U_{(5)}} = .2 \quad \frac{U_{(2)}}{U_{(5)}} = .4 \quad \frac{U_{(3)}}{U_{(5)}} = .6 \quad \frac{U_{(4)}}{U_{(5)}} = .8$$

Notice

$\frac{U_{(3)}}{U_{(5)}}$ is the 3^{rd} biggest decimal out of 4

i.e. $\frac{U_{(3)}}{U_{(5)}} = U_{(3)}$ out of 4

$$\sim \text{Beta}(3, \underbrace{4-3+1}_{=2})$$

More generally

let $U_{(1)}, \dots, U_{(m)}$ be m order statistics.

For $k \le \ell \le m$,

$$\frac{U_{(k)}}{U_{(\ell)}} = U_{(k)} \text{ out of } \ell-1$$

$$\sim \text{Beta}(k, \ell-1-k+1)$$

$$= \text{Beta}(k, \ell-k)$$

ex

Throw down 20 darts on $(0,1)$.

$Y = U_{(2)}$ $X = U_{(4)}$

a) what distribution is $\frac{Y}{X}$?

b) Find $P(X > 4Y)$

$$\frac{Y}{X} = \frac{U_{(2)} \text{ out of } 20}{U_{(4)} \text{ out of } 20} = U_{(2)} \text{ out of } 3$$

$$\sim \text{Beta}(2, \underbrace{3-2+1}_{=2})$$

$$P(X > 4Y) = P\left(\frac{Y}{X} < \frac{1}{4}\right)$$

$$Z = \frac{Y}{X} \sim \text{Beta}(2,2)$$

$$f_Z(z) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} z(1-z) = \boxed{6z(1-z)}$$

$$P(Z < 1/4) = 6\int_0^{1/4} z\,dz - 6\int_0^{1/4} z^2\,dz$$

$$= 10/64$$

Appendix

Iterated Expectation

We show $E(Y) = E(E(Y|X))$:

$$E(Y) = \sum_{\text{all } y} y P(Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(X=x, Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} \frac{P(X=x, Y=y)}{P(X=x)} P(X=x)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(Y=y|X=x) P(X=x)$$

$$= \sum \quad \sum y P(Y=y|X=x) \cdot P(X=x)$$

$$= \sum_{\text{all } x} E(Y|X=x) \cdot P(X=x)$$

$$= E(E(Y|X))$$

$\square$