Stat 134 lec 34

warmup: 10:00–10:10

Let $N \sim \text{Geom}(p)$ on $1, 2, 3, \ldots$

Suppose $X \mid N=n \sim \text{Unif}(0, 1, 2, \ldots, n)$

Find $E(X)$ Hint $E(X \mid N) = \frac{N}{2}$

Best way

$$E(X) = E(\underbrace{E(X \mid N)}_{=\frac{N}{2}}) = E\left(\frac{N}{2}\right) = \frac{1}{2}\underbrace{E(N)}_{=\frac{1}{p}} = \boxed{\frac{1}{2p}}$$

Alternate way

$$E(X) = \sum_{n=1}^{\infty} \underbrace{E(X \mid N=n)}_{=\frac{n}{2}} \underbrace{P(N=n)}_{=q^{n-1}p} = \frac{p}{2} \underbrace{\sum_{n=1}^{\infty} n q^{n-1}}_{=1/p^2 \text{ since}} = \boxed{\frac{1}{2p}}$$

$$\underbrace{E(N)}_{=\frac{1}{p}} = \sum_{n=1}^{\infty} n p q^{n-1} = p \sum_{n=1}^{\infty} n q^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} n q^{n-1} = \frac{1}{p^2}$$

Announcement:

Schedule next week!

M: regular lecture

W: MT2 review

F: regular lecture (MT2 available Friday 6pm - due Sunday 6pm)

Last time

Sec 6.2 Rule of iterated expectation

For any random variable T with finite expectation and any discrete RV S,

$$E(T) = E(E(T|S)) = \sum_{\text{all } s} E(T|S=s) \cdot P(S=s)$$

Today

1. Sec 6.2 Properties of conditional expectation
2. Sec 6.3 Conditional density
3. Sec 6.3 Bayesian statistics

1 Sec 6.2 Properties of conditional expectation

$(Y+Z)|X=x \;=\; Y|X=x \;+\; Z|X=x$ so

$E(Y+Z|X=x) = E(Y|X=x) + E(Z|X=x)$

What is $E(X+Z|X=5) = ?$

$\underbrace{E(X|X=5)}_{=5} + E(Z|X=5)$

Properties

1. $E(X) = E(E(X|Y))$ equality of numbers
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$

equality of RVs

Notation

If (X,Y) is joint discrete

$P(Y|X=x)$ is conditional prob,

If (X,Y) is joint continuous

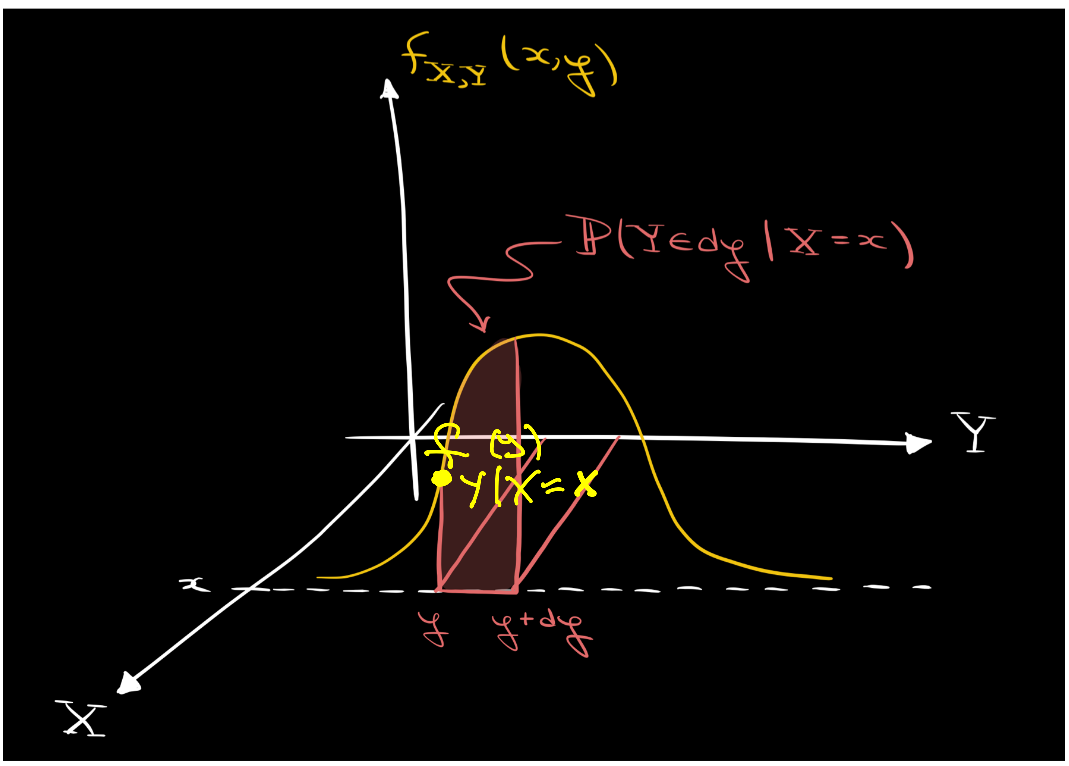
$f_{Y|X=x}(y)$ is conditional density.

② sec 6.3 Conditional Density:

Let X, Y be continuous RVs with joint density $f_{X,Y}(x,y)$

Let $f_{Y|X=x}(y)$ be a slice of $f_{X,Y}(x,y)$ through $X=x$.

Define $P(Y \in dy \mid X=x)$ as the area under $f_{Y|X=x}(y)$ for $Y \in dy$



By Baye's rule,

$$P(Y\in dY \mid X=x) = \lim_{dx\to 0} \frac{\overbrace{P(Y\in dY, X\in dX)}^{\approx f(x,y)\,dx\,dy}}{\underbrace{P(X\in dX)}_{\approx f_X(x)\,dx}}$$

$$\parallel$$

$$f_{Y|X=x}(y)\,dy$$

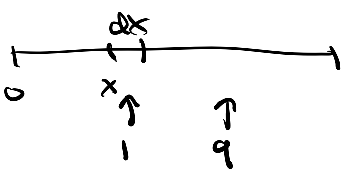
$$\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$$

Conditional density of Y given $X=x$

ex Let $U_1,\dots,U_{10} \overset{iid}{\sim} U(0,1)$

$X = U_{(1)}$, $Y = U_{(10)}$

$f(x,y) = 90\,(y-x)^8$



$$f_X(x) = \binom{10}{1,9}(1-x)^9 = 10\,(1-x)^9$$

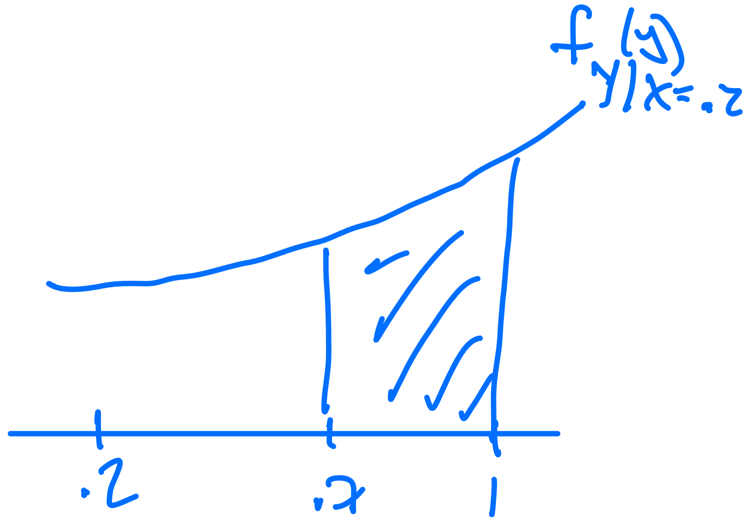
Find $P(Y > .7 \mid X = .2)$

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}$$

a) Find $f_{Y|X=.2}(y)$

$$f_{Y|X=.2}(y) = \frac{f(.2,y)}{f_X(.2)} = \boxed{\frac{90(y-.2)^8}{10(.8)^9}}$$

b) Find $\int_{.7}^{1} f_{Y|X=.2}(y)\,dy$



$$\frac{9}{(.8)^9}\int_{.7}^{1}(y-.2)^8\,dy = \frac{9}{(.8)^9}\int_{u=.5}^{u=.8} u^8\,du$$

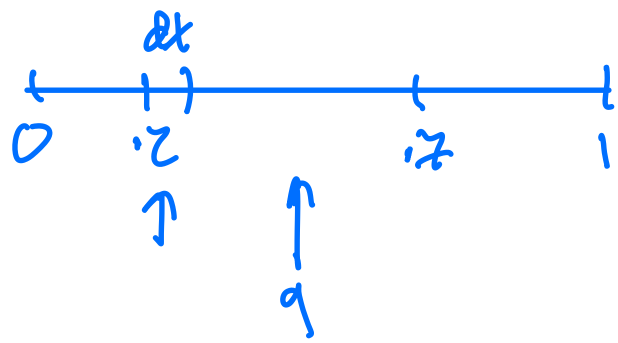
let $u = y-.2$

$$= \frac{9}{(.8)^9}\,\frac{u^9}{9}\Big|_{.5}^{.8} = \boxed{1-\left(\frac{.5}{.8}\right)^9}$$

Alternatively
use fact $x = U_{(1)}, y = U_{(10)}$

$$P(y \geq .7 \mid x = .2) = \lim_{dx \to 0} \frac{P(y \geq .7,\ x \in .2 + dx)}{P(x \in .2 + dx)}$$

$$= 1 - \lim_{dx \to 0} \frac{P(y \leq .7,\ x \in .2 + dx)}{P(x \in .2 + dx)}$$



$$P(y \leq .7,\ x \in .2 + dx) = \binom{10}{1,9} 1 dx (.7 - .2)^9$$

$$P(x \in .2 + dx) = \binom{10}{1,9} 1 dx (1 - .2)^9$$

$$1 - \lim_{dx \to 0} \frac{10 \cancel{dx} (.5)^9}{10 \cancel{dx} (.8)^9} = \boxed{1 - \left(\frac{.5}{.8}\right)^9}$$

Rule of average conditional probabilities (discrete case)

Let X and Y be discrete RVs w/ joint distribution $P(X=x, Y=y)$

$$P(Y=y) = \sum_x P(Y=y, X=x) = \sum_x P(Y=y \mid X=x)P(X=x)$$

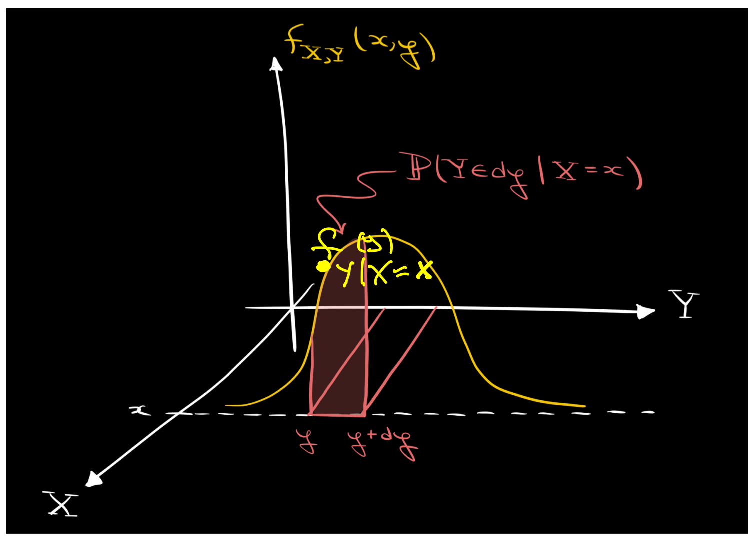
Rule of average conditional probabilities (Continuous case)

Let X and Y be continuous RVs w/ joint distribution $f(x,y)$

$$P(Y\in dy) = \int_{x\in X} P(Y\in dy, X=x)dx$$

$$= \int_{x\in X} P(Y\in dy \mid X=x) f_X(x)dx$$

$$= \int_{x\in X} f_{Y|X=x}(y)dy\, f_X(x)dx$$



ex $X \sim \text{Unif}(0,1)$

$I_1 | X=x,\ I_2 | X=x \overset{iid}{\sim} \text{Ber}(x)$

a) Find $P(I_2=1) = \int_{x=0}^{x=1} \overset{x}{\overset{\shortparallel}{P(I_2=1 \mid X=x)}}\, \overset{1}{\overset{\shortparallel}{f_X(x)}}dx$

$$= \int_{x=0}^{x=1} x\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{1/2}$$

similarly $P(I_2=1) = 1/2$

b) Find $P(I_2=1 \mid I_1=1)$

$$= \frac{P(I_2=1, I_1=1)}{P(I_1=1)}$$

$P(I_2=1 \mid X=x)\,P(I_1=1 \mid X=x) = x^2$

$$P(I_2=1, I_1=1) = \int_0^1 P(I_2=1, I_1=1 \mid X=x)\, f_X(x)\, dx$$

$$= \int_0^1 x^2\, dx = 1/3$$

$$P(I_2=1 \mid I_1=1) = \frac{1/3}{1/2} = \boxed{2/3}$$

Are I_1 I_2 independent?