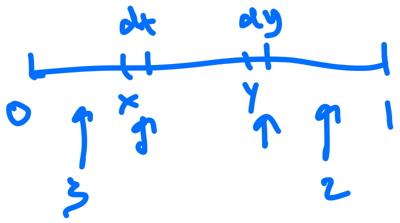
Stat 134 lec 36 (MT2 review)

Warmup 10:00-10:10

Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1-y)^2$ for $0 < x < y < 1$.

Fill in the blanks: X and Y represent the 4^{th} smallest and 5^{th} smallest of 7^{th} i.i.d. Unif (0,1) random variables, respectively.



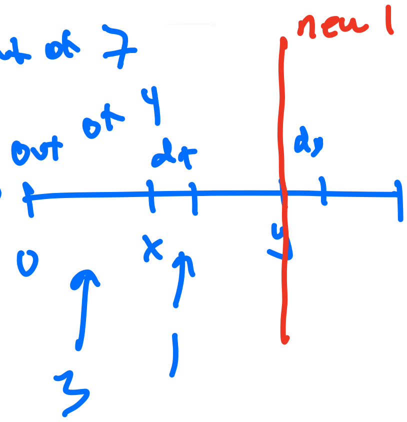
Let (X, Y) have joint density $f_{X,Y}(x, y) = 420x^3(1-y)^2$ for $0 < x < y < 1$.

(a) Find $P(3X < Y)$;

$X \sim U_{(4)}$ out of 7

$Y \sim U_{(5)}$ out of 7

$\frac{X}{Y} = \frac{U_{(4)}}{U_{(5)}} = U_{(4)}$ out of 4



$$= P\left(\frac{X}{Y} < \frac{1}{3}\right)$$

$$= P\left(U_{(4)} \text{ out of } 4 < \frac{1}{3}\right)$$

$$= \boxed{\left(\tfrac{1}{3}\right)^4}$$

ex

Let X, Y have joint density given by

$$f_{X,Y}(x,y) = \frac{\lambda}{y} e^{-\lambda y}, \quad 0 < x < y.$$

Find the marginal distribution of Y.

$$f_Y(y) = \int_{x=0}^{x=y} f_{X,Y}(x,y)\,dx$$

$$= \int_0^y \frac{\lambda}{y} e^{-\lambda y}\,dx$$

$$= \frac{\lambda e^{-\lambda y}}{y} x \Big|_{x=0}^{x=y}$$

$$= \lambda e^{-\lambda y}, \quad y > 0$$

$$= Y \sim \text{Exp}(\lambda).$$

(Change of variables, order statistics) Let $X \sim$ Uniform $(-1, 1)$ (this is a continuous uniform random variable).

$f_X(x) = \frac{1}{2}$ for $-1 < x < 1$

(a) Compute the density of $Y = e^X$.

Change of variable

$x = \ln y$ $\quad \frac{d\,g(x)}{dx} = e^x$

$$f_Y(y) = \frac{f_X(\ln(y))}{e^{\ln(y)}} = \frac{1}{2y} \text{ for } \frac{1}{e} < y < e$$

since

$-1 < \ln(y) < 1 \Leftrightarrow \frac{1}{e} < y < e$

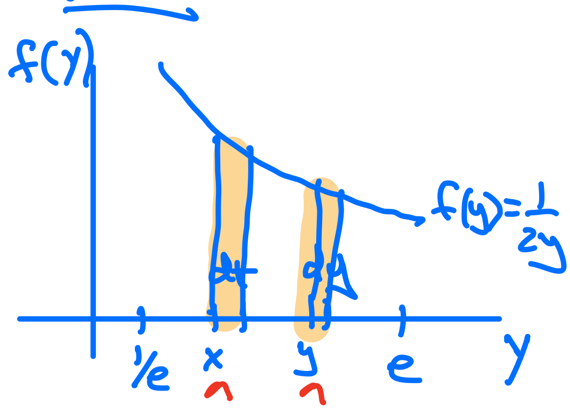
$\ln(y) = x$, $x \sim U(-1,1)$

(b) Let now X_1, X_2 be i.i.d. uniform random variables, and for each $i = 1, 2$, let $Y_i = e^{X_i}$. What is the joint density of $Y_{(1)}$ and $Y_{(2)}$, the minimum and the maximum of the Y_i's?

$$P(Y_{(1)} \in dx, Y_{(2)} \in dy) = f_{Y_{(1)}, Y_{(2)}}(x,y)\, dx\, dy$$

$$f_{Y_{(1)}, Y_{(2)}}(x,y) = \binom{2}{1,1} \frac{1}{2x} \cdot \frac{1}{2y}$$

Picture



Review MGF $M_X(t) = E(e^{tX})$

Main properties

① $M_X(0) = 1$

② $M_{aX}(t) = M_X(at)$

③ $M_X'(0) = E(X)$

$M_X''(0) = E(X^2)$

$\vdots$

$M_X^{(k)}(0) = E(X^{(k)})$

④ If $X_1, \dots, X_n$ are independent then

$M_{X_1 + \dots + X_n}(t) = M_{X_1}(t) \cdots M_{X_n}(t)$

⑤ $M_X(t)$ is unique for t in a neighborhood of 0. So if $M_X(t) = e^{t^2/2}$, for t around 0, then $X \sim N(0,1)$.

Taylor series around 0:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

ex

CLT

Let $X_1, \dots, X_n \overset{iid}{\sim} F$, mean μ, SD σ ← any distribution

$S_n = \sum_{i=1}^{n} X_i$

$S_n \longrightarrow N(n\mu, n\sigma^2)$ as $n \to \infty$

Pf/ We show that

$$\frac{S_n - n\mu}{\sqrt{n}\,\sigma} \longrightarrow Z \sim N(0,1) \text{ as } n \to \infty$$

Let $Y_i = \frac{X_i - \mu}{\sigma}$

$$\sum_{i=1}^{n} Y_i = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma}$$

($\sum_{i=1}^{n} X_i$ ← S_n)

So $\sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}} = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}$

We will show that for n large,

$\sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}$ and Z have the same MGF.

Note that

$$E(Y_i) = E\left(\frac{X_i - \mu}{\sigma}\right) = \frac{1}{\sigma} E(X_i - \mu) = 0$$

$$Var(Y_i) = \frac{1}{\sigma^2} Var(X_i - \mu) = \frac{1}{\sigma^2} \cdot \sigma^2 = 1$$

So $E(Y_i^2) = Var(Y_i) + E(Y_i)^2 = 1$

Make a Taylor series of $M_{\frac{Y_i}{\sqrt{n}}}(t)$ around 0:

$$M_{\frac{Y_i}{\sqrt{n}}}(t) = M_{Y_i}\left(\frac{t}{\sqrt{n}}\right)$$

(property ②.)

$$\frac{d}{dt} M_{\frac{Y_i}{\sqrt{n}}}(t) = \frac{d}{dt} M_{Y_i}\left(\frac{t}{\sqrt{n}}\right) = M'_{Y_i}\left(\frac{t}{\sqrt{n}}\right)\cdot\frac{1}{\sqrt{n}}$$

$$\frac{d^2}{dt^2} M_{\frac{Y_i}{\sqrt{n}}}(t) = M''_{Y_i}\left(\frac{t}{\sqrt{n}}\right)\cdot\frac{1}{n}$$

$$\frac{d^3}{dt^3} M_{\frac{Y_i}{\sqrt{n}}}(t) = M'''_{Y_i}\left(\frac{t}{\sqrt{n}}\right)\cdot\frac{1}{n^{3/2}}$$

$\vdots$

$$M_{\frac{Y_i}{\sqrt{n}}}(t) = M_{Y_i}(0) + M'_{Y_i}(0)\frac{t}{\sqrt{n}} + \frac{M''_{Y_i}(0)\frac{t^2}{n}}{2!} + \frac{M'''(0)\frac{t^3}{n^{3/2}}}{3!} + \dots$$

$$= 1 + E(Y_i)\frac{t}{\sqrt{n}} + \frac{E(Y_i^2)}{2!}\frac{t^2}{n} + \frac{E(Y_i^3)}{3!\,n^{3/2}} + \dots$$

($E(Y_i) = 0$, $E(Y_i^2) = 1$)

$$= 1 + 0 + \frac{1}{n}\left[\frac{t^2}{2} + \frac{t^3 E(\dot{Y}_i^3)}{3!\, n^{1/2}} + \cdots\right]$$

Note $\left[\frac{t^2}{2} + \frac{t^3 E(\dot{Y}_i^3)}{3!\, n^{1/2}} + \cdots\right] \approx \frac{t^2}{2}$ for large n

so

$$M_{\frac{Y_1}{\sqrt{n}}}(t) \approx 1 + \frac{1}{n}\frac{t^2}{2} \quad \text{for large } n$$

$\frac{Y_i}{\sqrt{n}}$ are independent,

$$M_{\frac{S_n - n\mu}{\sqrt{n}\sigma}}(t) = M_{\frac{Y_1}{\sqrt{n}}}(t) \cdots M_{\frac{Y_n}{\sqrt{n}}}(t)$$

$\left(1 + \frac{x}{n}\right)^n \approx e^x$ for large n

$$\frac{S_n - n\mu}{\sqrt{n}\sigma} = \sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}}$$

$$\rightarrow \left[1 + \frac{1}{n}\frac{t^2}{2}\right]^n \approx e^{\frac{t^2}{2}}$$

which is MGF of $N(0,1)$

Hence $\frac{S_n - n\mu}{\sqrt{n}\sigma} \rightarrow N(0,1)$

$\square$

(Question 6: = Joint Density, Convolution Let X, Y have joint density $f_{X,Y}(x,y) = 6\lambda^2 e^{-\lambda(x+y)}(1 - e^{-\lambda x/2})$ if $0 < x < 2y$ and $f_{X,Y}(x,y) = 0$ otherwise where $\lambda > 0$.

Find the density of $X + Y$.

Answer. Recall the convolution formula: if $Z = X + Y$, then

$$f_Z(z) = \int_0^z f_{X,Y}(x, z-x)dx = 6\lambda^2 \int_0^{2z/3} (1-e^{-\lambda x/2})e^{-\lambda z}dx = 4\lambda^2 ze^{-\lambda z} - 12\lambda e^{-\lambda z} + 12\lambda e^{-4\lambda z/3}$$

Gamma

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

a) Fill in the blank with a number: The fifth male traveler is expected to arrive at the desk _______ minutes after the first male traveler.

$$T \sim \text{Pois}\left(15 \cdot \frac{1}{60}\right)$$

$$M \sim \text{Pois}\left(\tfrac{1}{4} \cdot (.6)\right) = \text{Pois}\left(\tfrac{9}{60}\right)$$

$$F \sim \text{Pois}\left(\tfrac{1}{4} \cdot (.4)\right) = \text{Pois}\left(\tfrac{6}{60}\right)$$

T_5 = wait time of 5^{th} male $\sim \text{Gamma}(5, \tfrac{9}{60})$

T_1 = " " " 1^{st} male $\sim \text{Gamma}(1, \tfrac{9}{60})$

$$E(T_5 - T_1) = E(T_5) - E(T_1) = \frac{4 \cdot 60}{9} = \frac{4}{9/60}$$

$$E(T_5 - T_1) = E(T_4), \quad E(T_5) = \frac{5}{9/60}, \quad E(T_1) = \frac{1}{9/60}$$

$$= \boxed{\frac{80}{3} \text{ min}}$$

Note

$T_{m+n} - T_n = T_m$ for m, n positive integers. The difference of wait times is a wait time.

5. Travelers arrive at an airport Information desk according to a Poisson process at the rate of 15 per hour. Assume that each traveler arriving at the desk has a 60% chance of being male and a 40% chance of being female, independent of all other travelers.

b) Find the chance that the fifth male traveler arrives at the desk more than 30 minutes after the first male traveler.

$M \sim Pois(9/60)$
$W \sim Pois(6/60)$

$$P(W_5 - W_1 > 30) = P(W_4 > 30)$$
$$= P(N_{30} \leq 3)$$
$$= e^{-4.5}\left(1 + 4.5 + \frac{4.5^2}{2!} + \frac{4.5^3}{3!}\right)$$

since $N_{30} \sim Pois\left(\frac{9}{60} \cdot 30\right) = Pois(4.5)$