Stat 134 lec 40

warmup 10:00–10:10

let X, Y are std. bivariate normal, $\rho > 0$

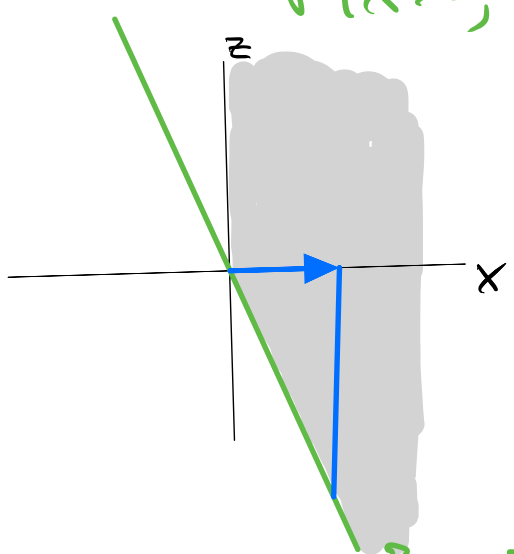
Find $P(X>0, Y>0)$

Hint

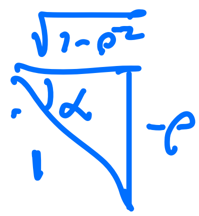
$$P(X>0, Y>0) = P(X>0, \rho X + \sqrt{1-\rho^2}\, Z > 0)$$

$$= P\left(X>0, Z > \frac{-\rho}{\sqrt{1-\rho^2}} X\right)$$

→ note X and Z are uncorrelated so joint (X,Z) is a bell over x, z with rotational symmetry.



$$z = \frac{-\rho}{\sqrt{1-\rho^2}} x$$



$$\tan\alpha = \frac{-\rho}{\sqrt{1-\rho^2}} \Rightarrow \alpha = \tan^{-1}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)$$

$$P\left(X>0, Z> \frac{-\rho}{\sqrt{1-\rho^2}} X\right) = \boxed{\frac{90 + |\alpha|}{360}}$$

last time

RRR week schedule

M Review (post questions on b-courses),

W OFF

F Review (post questions on b-courses)

OH after class in SLC

Sec 6.5. Bivariate Normal

Def^n (Standard Bivariate Normal Distribution)

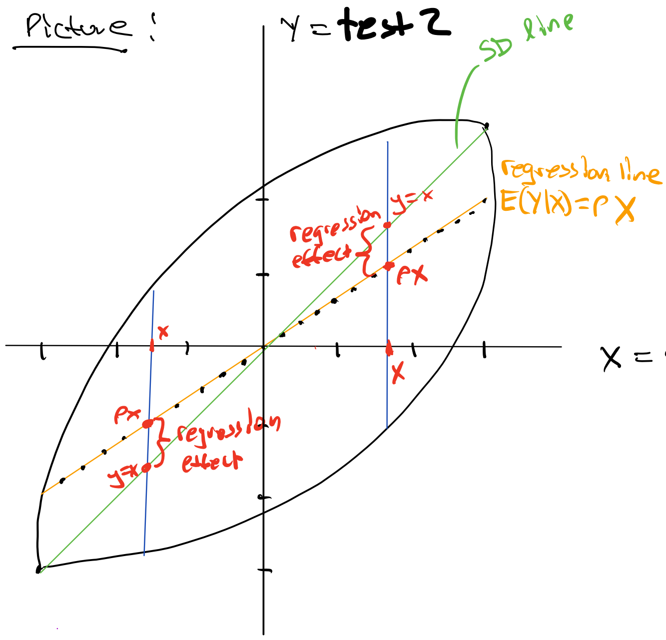
let X, Z iid $N(0,1)$, $-1 \le \rho \le 1$

$$Y = \rho X + \sqrt{1-\rho^2}\, Z \sim N(0,1)$$

$$\text{Corr}(X,Y) = \rho$$

regression effect

Picture:



Regression effect,

Corr (test 1, test 2) = .6

If 1 SD above mean on test 1 then on average you will be less than 1 SD above average on test 2.

(regression line is less steep than SD line).

Today ① MGF of bivariate normal

② Sec 6.5 properties of bivariate normal

① MGF of bivariate normal

The single and bivariate MGF is defined as:

$$M_Y(t) = E\left(e^{tY}\right)$$ single variable MGF

$$M_{(X,Y)}(s,t) = E\left(e^{sX+tY}\right)$$ bivariate MGF

Show that $M_{(X,Y)}(s,t) = M_X(s)\, M_Y(t)$ iff

X, Y are independent.

$$M_{(X,Y)}(s,t) = E\left(e^{sX+tY}\right)$$
$$= E\left(e^{sX} \cdot e^{tY}\right)$$
iff X, Y indep.
$$= E\left(e^{sX}\right) E\left(e^{tY}\right)$$
$$= M_X(s) \cdot M_Y(t)$$

$E(AB) = E(A)E(B)$
iff A, B are indep.

Here
$A = e^{sX}$
$B = e^{tY}$

Thm Let (X,Y) be standard bivariate normal.
The MGF of (X,Y) is

$$M_{(X,Y)}(s,t) = e^{\frac{s^2}{2}+\frac{t^2}{2}+st\rho}$$

Pf/
$$M_{X,Y}(s,t) = E\left[e^{sX+tY}\right]$$
$$= E\left[e^{sX+t(\rho X+\sqrt{1-\rho^2}Z)}\right]$$
$$= E\left[e^{(s+t\rho)X}\cdot e^{t\sqrt{1-\rho^2}Z}\right]$$
$$\overset{\text{independence}}{=} E\left[e^{(s+t\rho)X}\right]E\left[e^{t\sqrt{1-\rho^2}Z}\right]$$
$$= M_X(s+t\rho)\cdot M_Z(t\sqrt{1-\rho^2})$$

Recall that $X \sim N(0,1)$ so $M_X(a) = e^{\frac{a^2}{2}}$

Finish proof.

$$= e^{\frac{(s+t\rho)^2}{2}}\cdot e^{\frac{(t\sqrt{1-\rho^2})^2}{2}}$$
$$= e^{s^2/2+st\rho+\cancel{\frac{t^2\rho^2}{2}}}\cdot e^{\frac{t^2}{2}-\cancel{\frac{t^2\rho^2}{2}}}$$
$$= \boxed{e^{s^2/2+t^2/2+st\rho}}$$

② Properties of Bivariate Normal

Recall that if 2 RVs X, Y are independent, $Cov(X,Y)=0$

$\Rightarrow Corr(X,Y)=0$

However the converse isn't true in general. ($Corr(X,Y)=0 \not\Rightarrow X,Y$ indep.)

Let $X \sim N(0,1)$

$Y = X^2$

X and Y are dependent

We show X, Y are uncorrelated.

$$Cov(X,Y) = \underset{\overset{\|}{E(X^3)}}{E(XY)} - \underset{\overset{\|}{0}}{E(X)}E(Y)$$

Real world example

X = height

$Y = (\text{height})^2$

If you make a scatter plot of height vs $(\text{height})^2$ it will look uncorrelated even though height and $(\text{height})^2$ are dependent

Find $E(X^3)$

$$= \int_{-\infty}^{\infty} x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0$$

since $\frac{x^3}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is an odd function

or

$$M_X(t) = e^{t^2/2}$$

$$M_X'''(0) = te^{t^2/2} + 2te^{t^2/2} + t^3 e^{t^2/2} \Big|_{t=0} = 0$$

$\Rightarrow E(X^3) = 0$

$\Rightarrow Corr(X,Y) = 0$

Thm If (X,Y) is std bivariate normal then

$\rho = \mathrm{Corr}(X,Y) = 0$ iff X, Y are independent.

Pf/ From earlier we know

for any joint distribution (X,Y),

$M_{(X,Y)}(s,t) = m_X(s)\, m_Y(t)$ iff X, Y are independent.

Since (X,Y) is std bivariate normal,

$$M_{(X,Y)}(s,t) = e^{s^2/2 + t^2/2 + st\rho}$$

X, Y indep $\quad M_{(X,Y)}(s,t) = M_X(s)\, M_Y(t) \Rightarrow e^{st\rho} = 1 \Rightarrow \boxed{\rho = 0}$

$$e^{s^2/2 + t^2/2 + st\rho} = e^{s^2/2} \cdot e^{t^2/2}$$

Conversely, if $\rho = 0$

$$M_{(X,Y)}(s,t) = e^{s^2/2 + t^2/2 + st\rho} = e^{s^2/2 + t^2/2} = e^{s^2/2} \cdot e^{t^2/2} = M_X(s)\, M_Y(t) \Rightarrow X, Y \text{ indep.}$$

Recall from lec 30 that the sum of independent normal random variables is normal.

What about dependent normal random variables?

recall,

$M_Y(t) = E(e^{tY}) = M_{tY}(1)$ single variable MGF

$M_{(X,Y)}(s,t) = E(e^{sX+tY}) = M_{sX+tY}(1)$ bivariable MGF

recall,

$Z \sim N(\mu,\sigma^2)$ iff $M_Z(a) = e^{\mu a} e^{\sigma^2 \frac{a^2}{2}}$

Thm Let $X, Y \sim N(0,1)$ and $\mathrm{corr}(X,Y) = \rho$.
(X,Y) is std bivariate normal iff
$sX + tY$ is normal for all constants s,t.

Pf/

$\Rightarrow$: Suppose X, Y std bivariate normal.
(i.e $Y = \rho X + \sqrt{1-\rho^2}\, Z$ for X, Z iid $N(0,1)$)

$$sX + tY = sX + t(\rho X + \sqrt{1-\rho^2}\, Z)$$
$$= (s + t\rho)X + t\sqrt{1-\rho^2}\, Z$$

is normal since X, Z are indep normals

$\Leftarrow$: Suppose $sX+tY$ is normal,

note $E(sX+tY) = sE(X) + tE(Y) = 0$ (with $E(X)=0$, $E(Y)=0$)

$Var(sX+tY) = Var(sX) + Var(tY) + 2\,Cov(sX, tY)$

$= s^2 + t^2 + 2st\rho$

$2\,Cov(sX,tY) = 2st\,Cov(X,Y)$, $Cov(X,Y) = \rho$

Hence, $sX+tY \sim N(0, s^2+t^2+2st\rho)$

Recall,

For $Z \sim N(\mu, \sigma^2) \Rightarrow M_Z(a) = e^{\mu a} e^{\sigma^2 \frac{a^2}{2}}$

Then $M_{(X,Y)}(s,t) = M_{sX+tY}(1) = e^{(s^2+t^2+2st\rho)\frac{1^2}{2}}$

$sX+tY$ normal; σ^2

$= e^{\frac{s^2}{2} + \frac{t^2}{2} + st\rho}$ ← MGF of (X,Y)

$\Rightarrow (X,Y)$ is standard bivariate normal $\square$

We dont need to restrict ourselves to $X, Y \sim N(0,1)$

Corollary – Proof at end of lecture

Let $U \sim N(\mu_U, \sigma_U^2)$

$V \sim N(\mu_V, \sigma_V^2)$

and $Corr(U,V) = \rho$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff $sU + tV$ is normal for all constants s, t.

Appendix

Corollary

Let $U \sim N(\mu_U, \sigma_U^2)$
$V \sim N(\mu_V, \sigma_V^2)$
and $\mathrm{Corr}(U,V) = \rho$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff
$sU + tV$ is normal for all constants s,t.

Pf/ let $X = \dfrac{U-\mu_U}{\sigma_U}$ $\quad (U = \mu_U + X\sigma_U)$

$Y = \dfrac{V-\mu_V}{\sigma_V}$ $\quad (V = \mu_V + Y\sigma_V)$

$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$
iff $(X,Y) \sim BV(0,0,1,1,\rho)$

$$sU + ty = s(\mu_U + X\sigma_U) + t(\mu_V + y\sigma_V)$$
$$= \underbrace{s\sigma_U}_{a} X + \underbrace{t\sigma_V}_{b} y + \underbrace{sx\sigma_U + ty\sigma_V}_{\text{constant}}$$

this is normal for all constants s,t
iff $aX + by$ is normal for all constants a,b.

$\square$