Stat 134 lec 41 final review pt 1

Warmup 10:00 – 10:10

Stat 134

Friday May 3 2019

1. Let (X,Y) be bivariate normal. Then (2X+3Y+4, 6X-Y-4) is bivariate normal.

   a true

   b false

   c not enough info to decide

Show

$a(2X+3Y+4) + b(6X-Y-4)$ is normal for all a, b,

$= \underbrace{(2a+6b)X + (3a-b)Y}_{\text{normal since } (X,Y) \text{ is BVN}} + \underbrace{4a-4b}_{\text{constant}}$ is normal

$\Rightarrow (2X+3Y+4,\ 6X-Y-4)$ is BVN

Announcement:
Review materials are on stat134.org website.

Last time

(*) — in the appendix I give an example of two standard normals whose joint is not bivariate normal

Thm Let $X, Y \sim N(0,1)$ and $\text{corr}(X,Y) = \rho$.
(X,Y) is std bivariate normal iff
$sX + tY$ is normal for all constants s, t.

We dont need to restrict ourselves to $X, Y \sim N(0,1)$

Corollary

Let $U \sim N(\mu_U, \sigma_U^2)$
$V \sim N(\mu_V, \sigma_V^2)$
and $\text{Corr}(U,V) = \rho$
$(U,V) \sim BV(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$ iff
$sU + tV$ is normal for all constants s, t.

ex

Let M be a students score on the midterm of a class and F the student's score on the final of the same class.

Suppose $(M, F) \sim BV(\underset{\mu_M}{70}, \underset{\mu_F}{65}, \underset{\sigma_M^2}{8^2}, \underset{\sigma_F^2}{10^2}, \underset{\rho}{0.6})$.

Find the chance that the student scores higher on the final than the midterm.

Hint $P(F > M) = P(F - M > 0)$

what distribution is $F - M$?

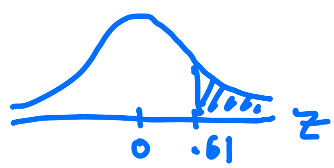
$F - M$ is normal since (M, F) is BVN

$E(F-M) = 65 - 70 = -5$

$\mathrm{Var}(F-M) = \mathrm{Var}(F) + \mathrm{Var}(M) - 2\underbrace{\mathrm{Cov}(F,M)}_{= \mathrm{Corr}(F,M)\,SD(M)\,SD(F)}$

$= 10^2 + 8^2 - 2(.6)(8)(10)$

$= \boxed{68}$



$\Rightarrow F - M \sim N(-5, 68)$

Make 0 into std units: $\dfrac{0 - E(F-M)}{SD(F-M)} = \dfrac{0-(-5)}{\sqrt{68}} = \boxed{.61}$

$P(F > M) = P(F - M > 0) = 1 - \Phi(.61) = \boxed{.27}$

Xinyi Liu

10:04am

Could we go over "exchangeable random variables" with more details and slowly (I think we went a little bit fast on this concept during lecture 38)? There's not much practice on this concept in our homework, so I don't have a specific question to ask, but could you maybe provide a sample question about this concept and go over it?

Also, could you explain more about the appendix part in lecture 38's notes about identically distributed distributed random variables not being exchangeable? Thanks!

Symmetry of a Joint Distribution

Let $X_1, \ldots, X_n$ be random variables with joint distribution defined by

$$P(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$$

The joint distribution is *symmetric* if $P(x_1, \ldots, x_n)$ is a symmetric function of $(x_1, \ldots, x_n)$. Equivalently, all $n!$ possible orderings of the random variables $X_1, \ldots, X_n$ have the same joint distribution. Then $X_1, \ldots, X_n$ are called *exchangeable*. Exchangeable random variables have the same distribution. For $2 \leq m \leq n$, every subset of m out of n exchangeable random variables has the same symmetric joint distribution of m variables.

$(X_1, X_3) = (X_2, X_5)$

Ex

52 card deck

draw 2 cards w/ replacement

X_1, X_2

X_1 = first draw, X_2 = second draw, we have

$$P(X_1 = J \text{ of } H, X_2 = Q \text{ of } S)$$
$$= P(X_1 = Q \text{ of } S, X_2 = J \text{ of } H)$$

In ~~fact~~ X_1 and X_2 are identically distributed (every card has probability $1/52$)

If the joint prob distribution is symmetric (i.e. $X_1, \dots, X_n$ is exchangable) then the marginal distributions are equal.

So $X_1, \dots, X_n$ exchangable $\Rightarrow$ $X_1, \dots, X_n$ identically distributed

however

$X_1, \dots, X_n$ identically distributed $\not\Rightarrow$ $X_1, \dots, X_n$ exchangable

Example of identically distributed RVs not exchangeable.

ex Flip coin 6 times

$q^2 p$

$I_2 = \begin{cases} 1 & \text{if } 2^{nd} \text{ flip is start of a run of 1 head} \\ 0 & \text{else} \end{cases}$

T H T _ _ _

I_2, I_3, I_4, I_5 are identically distributed,

Is joint of (I_2, I_3) the same as (I_2, I_5) ?

No

$P(I_2=1, I_3=1) = 0$ but $P(I_2=1, I_5=1) = P(I_2)P(I_5)$

$= (q^2 p)^2$

So I_2, I_3, I_4, I_5 are not exchangeable.

ex

7. Plant n apple seeds in a line. Each apple seed has a probability of p of growing into an apple tree, independent of other seeds. Let X equal the number of adjacent pairs of apple trees. (Note: If three adjacent apple seeds all grow into apple trees, count this as two adjacent pairs).

   (a) Write X as a sum of indicators. Are your indicators exchangeable?

   (b) Find E(X).

   (c) Find Var(X)

(a) Let I_i indicate an adjacent pair starts on the i^{th} trial. $X = I_1 + \cdots + I_{n-1}$. These indicators aren't exhangeable since the joints aren't all the same: for example $P(I_1 = 1, I_2 = 1) = p^3$ but $P(I_1 = 1, I_3 = 1) = p^4$.

(b) $E(I_1) = p^2$ so $E(X) = (n-1)E(I_1) = (n-1)p^2$.

$$I_2 = \begin{cases} 1 & p^2 \\ 0 & 1-p^2 \end{cases}$$

(c) $E(I_1 I_2) = p^3, E(I_1 I_3) = p^4$.

$$\begin{aligned} E(X^2) = \quad & (n-1)\overset{=p^2}{E(I_1)} + 2(n-2)E(I_1 I_2) + 2[1 + 2 + \cdots + (n-3)]E(I_1 I_3) \\ = \quad & (n-1)p^2 + 2(n-2)p^3 + (n-2)(n-3)p^4 \end{aligned}$$

$$Var(X) = E(X^2) - [E(X)]^2 = (n-1)p^2 + 2(n-2)p^3 + (n-2)(n-3)p^4$$

$\underline{n=5}$

	I_1	I_2	I_3	I_4
I_1	I_{11}	I_{12}	I_{13}	I_{14}
I_2	I_{21}	I_{22}	I_{23}	I_{24}
I_3	I_{31}	I_{32}	I_{33}	I_{34}
I_4	I_{41}	I_{42}	I_{43}	I_{44}

Ziyi Ding
8:01pm

Hi Adam, I am wonder if you can go over the difference between convergence in probability, convergence in distribution, and almost sure convergence. I think we briefly went over that in one of the sections.

This is beyond the scope of this class. In short when we say that a sequence of RVs $X_1, \dots, X_n$ converges to a RV X, this is more complicated than saying that a sequence of numbers converges to another number.

We talked in this class about convergence in distribution. This means $F_{X_n}(x) \to F_X(x)$ pointwise for all continuous points as $n \to \infty$

In Stat 205 you learn about convergence in probability and convergence almost surely. These are stronger forms of convergence.

Aaron Zaks
7:09pm

Could you cover the "meat and potatoes" of the course? just a rough review of the main concepts

A "main topic" we should review some more is writing expectation and variance in terms of conditional expectation and conditional variance.

Formulas:

$$E(X) = E(E(X|U))$$

$$Var(X) = E(Var(X|U)) + Var(E(X|U))$$

$\underline{ex}$ Let $U \sim Unif(0,1)$ and $X|U \sim Exp\left(\frac{1}{U}\right)$

Recall $E\left(Exp\left(\frac{1}{U}\right)\right) = U \Rightarrow E(X|U) = U$

$Var\left(Exp\left(\frac{1}{U}\right)\right) = U^2 \Rightarrow Var(X|U) = U^2$

Find $E(X)$ and $Var(X)$.

$$E(X) = E(E(X|U)) = E(U) = \boxed{1/2}$$

$$Var(X) = E(\underbrace{Var(X|U)}_{=U^2}) + Var(\underbrace{E(X|U)}_{=U})$$

$$E(U^2) = \underbrace{Var(U)}_{=1/12} + (\underbrace{E(U)}_{=(1/2)})^2 = \boxed{1/3}$$

$$Var(X) = 1/3 + 1/12 = \boxed{5/12}$$

Appendix

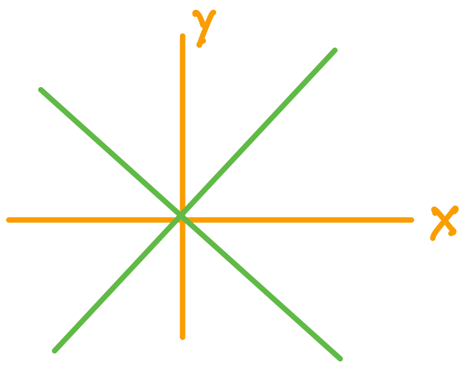
Most joint normals that come up in applications are bivariate normal but it is possible the joint of two normals is not bivariate normal.
Here is a simple example.

(Non-example of MVN). Here is an example of two r.v.s whose marginal distributions are Normal but whose joint distribution is not Bivariate Normal. Let $X \sim N(0, 1)$, and let

$$S = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

be a random sign independent of X. Then $Y = SX$ is a standard Normal r.v., due to the symmetry of the Normal distribution However, (X, Y) is not Bivariate Normal because $P(X + Y = 0) = P(S = -1) = 1/2$, which implies that $X + Y$ can't be Normal (or, for that matter, have any continuous distribution). Since $X + Y$ is a linear combination of X and Y that is not Normally distributed, (X, Y) is not Bivariate Normal.

Note that the scatter diagram of (X, Y) looks like



which is not the shape of a bivariate normal (football).
Also note that $X+Y$ isn't normal since $P(X+Y=0) = 1/2$ implies $X+Y$ is a mixed distribution.