Stat 134 lec 5

warm up 10:00 - 10:10

Stats 134

Chapter 2

1. Ten cards are dealt off the top of a well shuffled deck. The binominal formula doesn't apply to find the chance of getting exactly three diamonds because:

   unconditional

   a The probability of a trial being successful changes

   b The trials aren't independent

   c There isn't a fixed number of trials

   d more than one of the above

The (unconditional) probability of getting a diamond is 1/4. However the trials are dependent on one another.

If cards dealt with replacement
$X = \#$ diamonds in 10 trials
$P(X=3) = \binom{10}{3}(1/4)^3(3/4)^7$ binomial formula

Announcement: Quiz 1 covers Sec 1.1 — 1.6 and 2.1
3 or 4 questions in 45 minutes.
Section next week is in person as is the lecture.
Lecture will be recorded

Last time

Sec 2.1 The Binomial Distribution

A Bernoulli trial has 2 outcomes, success (Prob p)
and failure. (Prob $q = 1 - p$)

(think of tossing a coin having prob p of landing head)

n independent Bernoulli trials, each with prob p of success, has a Binomial distribution written $Bin(n,p)$.

For $k = 0, 1, 2, \ldots, n$ the Binomial formula says

$$P(k) = \binom{n}{k} p^k q^{n-k}$$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

the prob of getting exactly k out of n successes.

eg $n = 5$, $p = 1/2$

$P(0) = \binom{5}{0}(1/2)^0(1/2)^5 = 1/32$

$P(1) = \binom{5}{1}(1/2)^1(1/2)^4 = 5/32$

$P(2) = \binom{5}{2}(1/2)^2(1/2)^3 = 10/32$

$P(3) = 10/32$

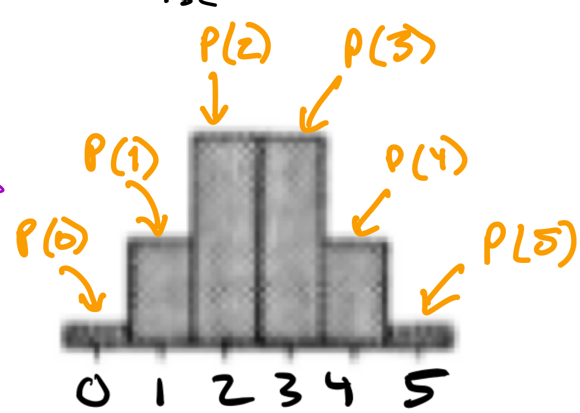
$P(4) = 5/32$

$P(5) = 1/32$

1	1	0	0	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1
0	1	1	0	0
0	1	0	1	0
0	1	0	0	1
0	0	1	1	0
0	0	1	0	1
0	0	0	1	1

number of arrangements of two 1s in 5 slots

$$\binom{5}{2} = \frac{5!}{2! \cdot 3!} = 10$$



ex

Halloween will be here before you know it and the children in your neighborhood will come trick-or-treating (that is, they will come to your door and demand candy). Suppose there are 20 children in your neighborhood and 30 houses (one of which is yours). Each child independently chooses 10 houses at random without replacement to visit.

a What is the probability that a specific child will visit your house? $p = 1/3$

b What is the probability that exactly 10 children visit your house?

$n = 20$
$k = 10$
$p = 1/3$

$$\boxed{P(X=10) = \binom{20}{10}\left(\frac{1}{3}\right)^{10}\left(\frac{2}{3}\right)^{10}}$$

Today

1. Finish sec 2.1 Binomial distribution
2. Start sec 2.2 Normal approximation to the binomial.

(1) Binomial Dist — The mode and mean

What does the binomial distribution look like for different n, p?

(most likely outcome)

Def^n (Mode)
The mode of the binomial distribution is the most likely outcomes (i.e. the k such that $P(k) = \binom{n}{k} p^k (1-p)^{n-k}$ is largest)

* ← Proof at end of notes using $\frac{P(k)}{P(k-1)}$, the consecutive odds ratio,

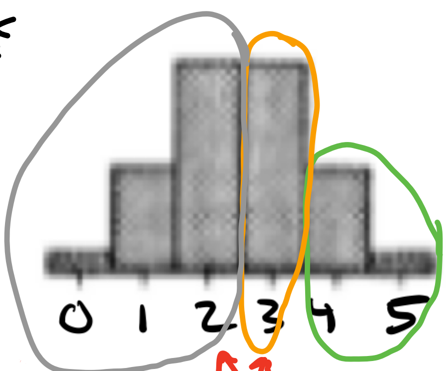
Thm For $k \in \{1, 2, \dots, n\}$

given by Binomial Formula,

- $k < np+p$ iff $P(k-1) < P(k)$
- $k > np+p$ iff $P(k-1) > P(k)$
- $k = np+p$ iff $P(k-1) = P(k)$

Picture

ex

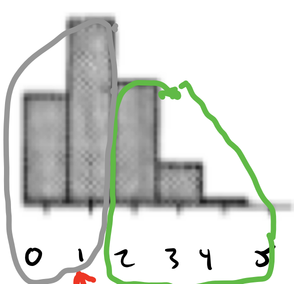


mode

$n=5$
$k = 1,2,3,4,5$
$p = 1/2$

$np+p = 5(1/2) + 1/2 = \boxed{3}$

ex



mode.

ex $n=5$, $p=1/4$, $k = 1,2,3,4,5$

$np+p = 5 \cdot \frac{1}{4} + \frac{1}{4} = \boxed{1.5}$

The mode for the binomial distribution has 2 cases:

integers.

$$\text{mode} = \begin{cases} m & \text{if } np+p \notin \mathbb{Z} \\ m-1, m & \text{if } np+p \in \mathbb{Z} \end{cases} \qquad \text{where } m = \lfloor np+p \rfloor$$

the integer part of $np+p$

The mode is a measure of the center of the data. However, the true center of your data is the <u>expectation</u> (i.e the <u>mean</u>).

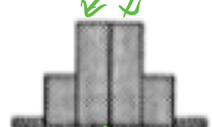
shown in chap 3

<u>Fact</u> The expected (mean) number of success is $\boxed{\mu = np}$

This isn't usually an integer

ex $n=5$, $p=1/2$ $\quad \mu = np = 5/2$

mode



mean = $5/2$

Given $0<p<1$, if the mean is an integer is it the mode?

Yes, $np \in \mathbb{Z} \Rightarrow np+p \notin \mathbb{Z} \Rightarrow$ there is a single mode $m = \lfloor np+p \rfloor = np$

the mean

shown in chap 3

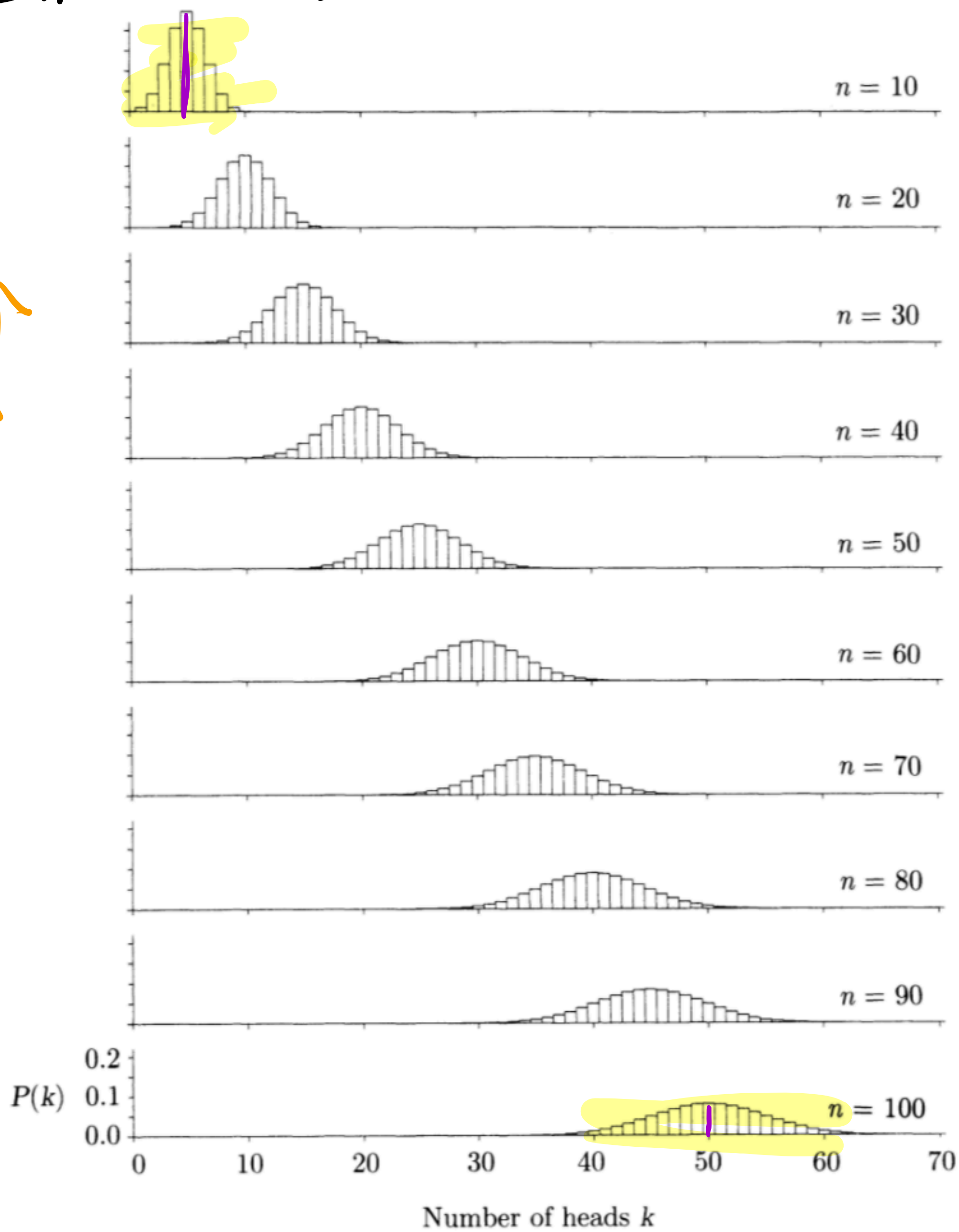
<u>Fact</u> the average spread around the mean (standard deviation) is $\sigma = \sqrt{npq}$ where $q = 1-p$

Notice that the spread around the mean gets larger as n gets bigger,

P88

Binomial $(n, \frac{1}{2})$ $\qquad \sigma = \sqrt{npq} = \sqrt{n \frac{1}{2} \cdot \frac{1}{2}} = \frac{\sqrt{n}}{2}$

Notice spread ↑ as n ↑



law of averages says

if X = # heads in n fair coin tosses

$\frac{X}{n} \to p = \frac{1}{2}$

$n=10 \quad P(5) = \binom{10}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^5 = \boxed{.246}$

$n=100 \quad P(50) = \binom{100}{50}\left(\frac{1}{2}\right)^{50}\left(\frac{1}{2}\right)^{50} \approx \boxed{.08}$

Stat 134

49900 heads

50100 heads

1. A fair coin is tossed, and you win a dollar if there are exactly 50% heads. Which is better?

   a 10 tosses

   b 100 tosses

② sec 2.2 The normal distribution

The normal curve is $y = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Notice :

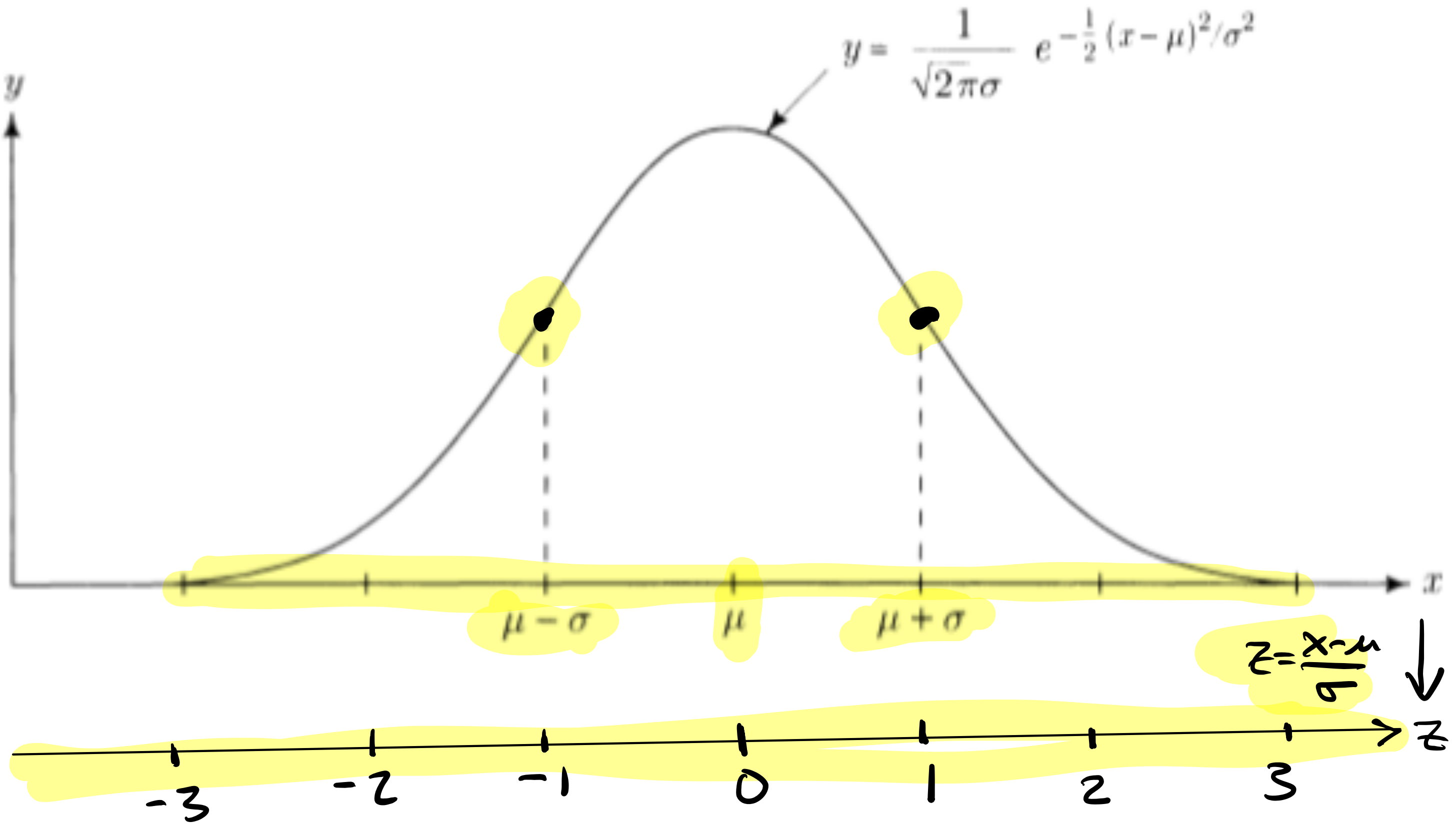
① two param μ = mean
σ = std dev

② inflection pts $\mu \pm \sigma$

③ almost all data between $\mu \pm 3\sigma$

p93

FIGURE 1. The normal curve.



To find the area under the curve it is convenient to make a change of coordinates

$$z = \frac{x-\mu}{\sigma}$$

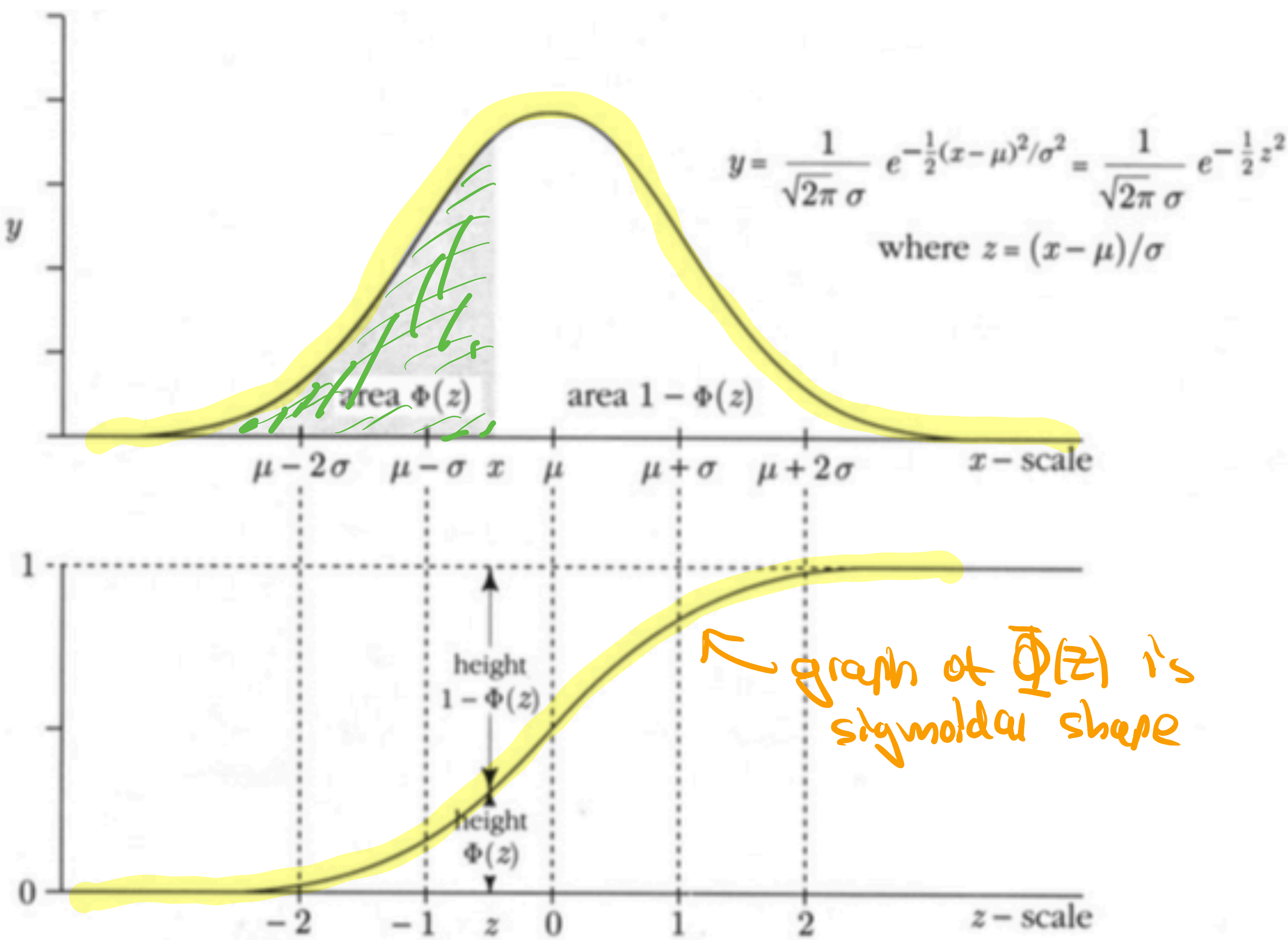
This makes $\mu = 0$ and $\sigma = 1$

density function

$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ std normal curve

Define cummutative distribution function (cdf) as $\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt$ ← area between $-\infty$ and z

P95

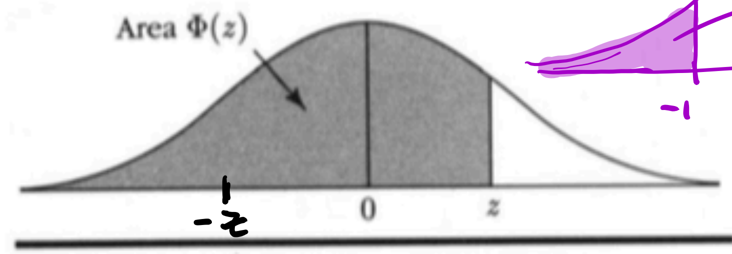


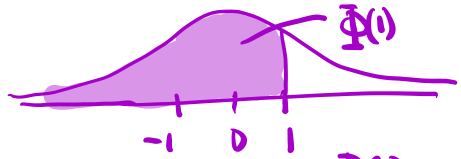
graph of $\Phi(z)$ is sigmoidal shape

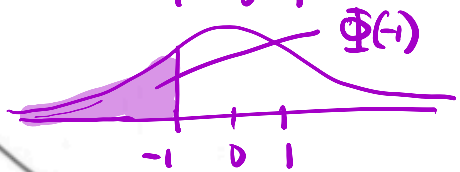
we cant solve integral $\Phi(z)$ but instead use a look up table.

Notice

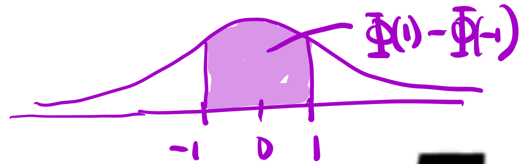
$\Phi(-z) = 1 - \Phi(z)$







⇒



Appendix 5

Normal Table

Table shows values of $\Phi(z)$ for z from 0 to 3.59 by steps of .01. Example: to find $\Phi(1.23)$, look in row 1.2 and column .03 to find $\Phi(1.2 + .03) = \Phi(1.23) = .8907$. Use $\Phi(z) = 1 - \Phi(-z)$ for negative z.

Find area between 1 and -1 in std normal curve:

$\Phi(1) - \Phi(-1)$

$= .8413 \quad (1 - .8413)$

$= 2(.8413) - 1$

$= \boxed{.68}$

Find area between 2 and -2.

$2\Phi(2) - 1$

$= 2(.9772) - 1 = .95$

Find area between 3 and -3.

.997

	.0	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998

This is known as the emphirical rule.

Appendix

Thm

① $K < np + p$ iff $P(K-1) < P(K)$

② $K > np + p$ iff $P(K-1) > P(K)$

③ $K = np + p$ iff $P(K-1) = P(K)$

Pf/

First note that
$$\frac{\binom{n}{K}}{\binom{n}{K-1}} = \frac{\frac{\cancel{n!}}{K!(n-K)!}}{\frac{\cancel{n!}}{(K-1)!(n-K+1)!}} = \boxed{\frac{n-K+1}{K}}$$

Called the consecutive odds ratio

binomial formula

$$\frac{P(K)}{P(K-1)} = \frac{\binom{n}{K} p^K (1-p)^{n-K}}{\binom{n}{K-1} p^{K-1} (1-p)^{n-K+1}} = \boxed{\frac{n-K+1}{K} \cdot \frac{p}{1-p}}$$

$$P(K-1) < P(K)$$

$$\iff 1 < \frac{P(K)}{P(K-1)}$$

$$\iff 1 < \frac{n-K+1}{K} \cdot \frac{p}{1-p}$$

$$\iff K(1-p) < (n-K+1)p$$

$$\iff K - \cancel{Kp} < np - \cancel{pK} + p$$

$$\iff K < np + p$$

so $P(K-1) < P(K) \iff K < np + p$

similarly for $>$ or $=$ instead of $<$ $\square$