Stat 134 lec 2

Warm up 1:00 - 1:10

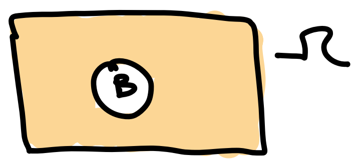
Prove the complement rule

$P(B^c) = 1 - P(B)$

Difference rule

$P(AB^c) = P(A) - P(AB)$

Picture



$\Omega = B \cup B^c$ disjoint union

$P(\Omega) = P(B) + P(B^c)$ addition rule

$\overset{\parallel}{1}$ (i.e. $P(\Omega) = 1$)

$\Rightarrow P(B^c) = 1 - P(B).$

or let $A = \Omega$

$\underset{P(B^c)}{\underset{\parallel}{P(\Omega B^c)}} = \underset{1}{\underset{\parallel}{P(\Omega)}} - \underset{P(B)}{\underset{\parallel}{P(\Omega B)}}$ difference rule

$\Rightarrow P(B^c) = 1 - P(B)$

unconditional probability.

Last time

Addition rule (OR) if A, B mutually exclusive sets

$$P(A \text{ or } B) = P(A) + P(B).$$

Inclusion exclusion (OR) $P(A \text{ or } B) = P(A) + P(B) - P(AB)$

Today

(0) Mathematical Induction

(1) Sec 1.3 Distributions

(2) Sec 1.4 Conditional Probability

(0) Mathematical Induction

A **proof by induction** consists of two cases. The first, the **base case** (or **basis**), proves the statement for $n = 0$ (or 1) without assuming any knowledge of other cases. The second case, the **induction step**, proves that *if* the statement holds for any given case $n = k$, *then* it must also hold for the next case $n = k + 1$. These two steps establish that the statement holds for every natural number n.

ex (1.3.12 in HW#1)

12. Inclusion–exclusion formula for n events. Derive the inclusion–exclusion formula for n events

$$P(\bigcup_{i=1}^{n} A_i) = \sum_i P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) - \cdots + (-1)^{n+1} P(A_1 \dots A_n)$$

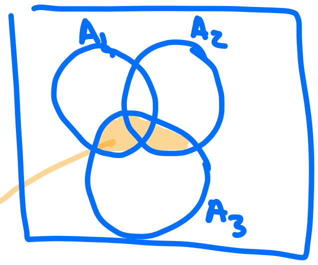
ex $n=3$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1A_2) - P(A_1A_3) - P(A_2A_3) + P(A_1A_2A_3)$$

Assume fact from set theory

$$\left(\bigcup_{i=1}^{k} A_i\right) A_{k+1} = \bigcup_{i=1}^{k} A_i A_{k+1} \quad (*)$$

ex $k=3$



$(A_1 \cup A_2) A_3 = A_1 A_3 \cup A_2 A_3$

To prove generalized inclusion exclusion for $n=3$ we show by induction.

True for $n=1$ $\quad P(A_1) = P(A_1)$ ✓

Assume true for $n=2$.

Show true for $n=3$:

$$P(A_1 \cup A_2 \cup A_3) =$$

$$P\left(\bigcup_{i=1}^{2} A_i \cup A_3\right) = P\left(\bigcup_{i=1}^{2} A_i\right) + P(A_3) - P\left(\left(\bigcup_{i=1}^{2} A_i\right) A_3\right)$$

$AB = A \cap B$

$$\left(\bigcup_{i=1}^{2} A_i\right) A_3 = \bigcup_{i=1}^{2} A_i A_3 \text{ by } (*)$$

$$= P\left(\bigcup_{i=1}^{2} A_i\right) + P(A_3) - P\left(\bigcup_{i=1}^{2} A_i A_3\right)$$

$$P\left(\bigcup_{i=1}^{2} A_i\right) = P(A_1) + P(A_2) - P(A_1 A_2)$$

$$P\left(\bigcup_{i=1}^{2} A_i A_3\right) = P(A_1 A_3) + P(A_2 A_3) - P(\underbrace{A_1 A_3 A_2 A_3}_{= A_1 A_2 A_3})$$

$$= P(A_1) + P(A_2) + P(A_3) - P(A_1 A_2) - P(A_1 A_3) - P(A_2 A_3) + P(A_1 A_2 A_3)$$

For HW,

try $n=4$ and generalize,

$$P\left(\bigcup_{i=1}^{3} A_i \cup A_4\right) = P\left(\bigcup_{i=1}^{3} A_i\right) + P(A_4) - P\left(\bigcup_{i=1}^{3} A_i A_4\right)$$

① **Sec 1.3 Distributions**

Uniform distribution

Let $\{x_1, x_2, \dots, x_n\}$ be a finite set.

Suppose the probability of drawing each element is equally likely (i.e each has prob $\frac{1}{n}$)

We say $\{x_1, \dots, x_n\}$ has the uniform distribution.

We write $\text{Unif}(\{x_1, \dots, x_n\})$.

ex $\{1, 1, 2\}$ is a finite set.

$\text{Unif}(\{1,1,2\})$ means 1 has probability $\frac{2}{3}$ and 2 has probability $\frac{1}{3}$.

ex $\overset{7}{\text{Suppose}}$ $\overset{1}{\text{a}}$ $\overset{4}{\text{word}}$ $\overset{2}{\text{is}}$ $\overset{8}{\text{randomly}}$ $\overset{6}{\text{picked}}$ $\overset{4}{\text{from}}$ $\overset{4}{\text{this}}$ $\overset{8}{\text{sentance}}$.

Name the distribution of the length of the word picked?

$\text{Uniform}(\{7, 1, 4, 2, 8, 6, 4, 4, 8\})$

length of word picked

$P(X=7) = 1/9$

$P(X=1) = 1/9$

$P(X=4) = 3/9$

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Stat 134

1. A deck of cards is shuffled. What is the chance that the top card is the **king** of spades **or** the bottom card is the **king** of spades

   P(AB)

   answer it with replacement,

   a $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{52}$

   b $\frac{1}{52} + \frac{1}{51}$

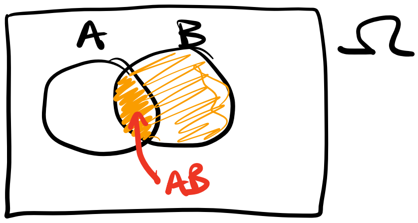
   c $\frac{1}{52} + \frac{1}{52} - \frac{1}{52} \times \frac{1}{51}$

   d none of the above

   $\frac{1}{52} + \frac{1}{52} = \frac{1}{2}$

② Sec 1.4 Conditional Probability and Independence

Let A, B be subsets of Ω (i.e events).



given

Baye's rule says $P(A|B) = \dfrac{P(AB)}{P(B)}$

$\iff \boxed{P(\underbrace{AB}_{\text{A and B}}) = P(A|B)P(B)}$ multiplication rule. (AND)

We say A and B are independent iff

$P(A|B) = P(A)$

or equivalently if $P(AB) = P(A)P(B)$

ex
A = last card is queen of spades
B = 1st card is king of spades

Is A and B independent?

$$P(AB) = \underset{1/52}{P(B)}\,\underset{1/51}{P(A|B)} \neq \underset{1/52}{P(B)}\,\underset{1/52}{P(A)}$$

$\Rightarrow$ A, B are dependent (i.e not independent) ex

(10 pts) An airport bus drops off 35 passengers at ~~7~~ 2 stops. Each passenger is equally likely to get off at any stop, and passengers act independently of one another. The bus makes a stop only if someone wants to get off. Find the probability that the bus drops off passengers at every stop.

Let B_i = event drop off at least one person at stop i.

$(AB)^c = A^c \cup B^c$

$$\begin{aligned}
P(B_1B_2) &= 1 - P\left((B_1B_2)^c\right) \\
&= 1 - P(B_1^c \cup B_2^c) \\
&= 1 - \left[P(B_1^c) + P(B_2^c) - \underbrace{P(B_1^cB_2^c)}_{=0}\right] \\
&= 1 - \left[\left(\tfrac{1}{2}\right)^{35} + \left(\tfrac{1}{2}\right)^{35}\right]
\end{aligned}$$

Now suppose 3 stops:

$$\begin{aligned}
P(B_1B_2B_3) &= 1 - P\left((B_1B_2B_3)^c\right) \\
&= 1 - P(B_1^c \cup B_2^c \cup B_3^c) \\
&= 1 - \sum_{i=1}^{3} P(B_i^c) + \sum_{i<j} P(B_i^cB_j^c) - \underbrace{P(B_1^cB_2^cB_3^c)}_{=0} \\
&= 1 - \left[3\left(\tfrac{2}{3}\right)^{35} + 3\left(\tfrac{1}{3}\right)^{35}\right]
\end{aligned}$$

$\binom{3}{1}$ since you choice 1 stop i out of 3 stops

$\binom{3}{2}$ since you choose 2 stops $i<j$ out of 3

If the bus has 7 stops:

$$P(B_1 B_2 \cdots B_7)^{\text{start}} = 1 - \binom{7}{1}\left(6/7\right)^{35} + \binom{7}{2}\left(5/7\right)^{35} - \cdots - \binom{7}{7}\left(\frac{7-7}{7}\right)^{35}$$

$$= \boxed{\sum_{j=0}^{7} (-1)^j \binom{7}{j}\left(\frac{7-j}{7}\right)^{35}}$$

Inclusion–exclusion formula for n events. Derive the inclusion–exclusion formula for n events

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) - \cdots + (-1)^{n+1} P(A_1 \ldots A_n)$$

$\binom{n}{1}$ terms $\binom{n}{2}$ terms $\binom{n}{3}$ terms $\binom{n}{n}$ terms