Stat 134 Lec 24

Warmup

Show that the MGF of $X \sim Pois(\mu)$ is

$M_X(t) = e^{\mu(e^t - 1)}$ for all t.

Recall $M_X(t) = E(e^{tX})$

and $P(X=k) = \frac{e^{-\mu}\mu^k}{k!}$

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} P(X=k) = \sum_{k=0}^{\infty} e^{tk} \frac{e^{-\mu}\mu^k}{k!}$$

$$= e^{-\mu} \sum_{k=0}^{\infty} \frac{e^{tk}\mu^k}{k!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(e^t\mu)^k}{k!}$$

$(e^t)^k$

Recall from Calculus

$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$ Taylor for all $a \in \mathbb{R}$

$$= e^{-\mu} e^{e^t \mu}$$

$$= \boxed{e^{\mu(e^t - 1)} \text{ for all } t}$$

Note

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = e^{\mu(e^t-1)} \cdot \mu e^t \Big|_{t=0} = \boxed{\mu}$$

✓

Announcement: Q3 next Thursday
covers Sec 4.1, 4.2, 4.4, 4.5, MGF

Last time

MGF

$$M_X(t) = E(e^{tX})$$

Thm If a MGF exists in an interval around zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

Today

1. Key properties of MGF
2. Recognizing a distribution from the variable part of its density

1) Key Properties of MGF

(a) If an MGF exists in an interval containing zero, $M^{(k)}(t)\Big|_{t=0} = E(X^k)$

last time

(b) If X and Y are independent RVs,
$M_{X+Y}(t) = M_X(t) M_Y(t)$.

Proved in MGF HW.

(c) If $M_X(t) = M_Y(t)$ for all t in an interval around 0 then $F_X(z) = F_Y(z)$
(i.e X and Y have the same distribution).

Skip proof — we can invert a MGF to get the CDF.

ex If $M_X(t) = \frac{1}{2}e^{1t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$ $\left(= E(e^{tX})\right)$,
e^{xt} tells us the value of X and the associated coefficients tell us the probabilities
(i.e X = 1, 2, 3 w/ prob ½, ⅓, ⅙.)

so MGF ⇒ distribution of X when X has finite # values.

Property (a) is useful to find E(X), Var(X),
Property (b) and (c) allow us to prove for example that sum of independent Poisson is Poisson.

ex

$X_1 \sim \text{Pois}(\mu_1)$
$X_2 \sim \text{Pois}(\mu_2)$ } independent.

Show that $X_1 + X_2 \sim \text{Pois}(\mu_1 + \mu_2)$.

$M_{X_1}(t) = e^{\mu_1(e^t - 1)}$ for all t

$M_{X_2}(t) = e^{\mu_2(e^t - 1)}$ for all t

$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = \boxed{e^{(\mu_1+\mu_2)(e^t - 1)}}$ for all t

MGF of Pois $(\mu_1 + \mu_2)$ for all t.

$\Rightarrow X_1 + X_2 \sim \text{Pois}(\mu_1 + \mu_2)$

ex Let X be a RV and a a constant.

Show that $M_{aX}(t) = M_X(at)$ $\leftarrow E(e^{Xat})$

hint $M_{aX}(t) = E(e^{aXt})$

$= E(e^{Xat})$

$= M_X(at)$

For $X \sim \text{Gamma}(r, \lambda)$

recall $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$

ex Let $X \sim \text{Exp}(\lambda)$ and $a > 0$.

Show that $Y = aX$ is also exponential, and specify the new parameter.

Note $M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^1$ for $s < \lambda$

$Y = aX$

for $at < \lambda$

$$M_{aX}(t) = M_X(at) = \left(\frac{\lambda}{\lambda - at}\right)^1 = \left(\frac{\frac{\lambda}{a}}{\frac{\lambda}{a} - t}\right)^1 \quad \text{for } t < \frac{\lambda}{a}$$

$\Rightarrow Y \sim \text{Exp}\left(\frac{\lambda}{a}\right)$

ex

r λ

$\shortparallel$ $\shortparallel$

(3 pts) Let X_i follow the Gamma (1/100, 2/100) distribution for $i = 1, 2 \ldots, 100$, independently of each other. We are interested in finding the distribution of the sample average, $Y = \frac{1}{100}\sum_{i=1}^{100} X_i$. Using properties of MGFs, identify the distribution of Y.

Recall that for $X \sim$ Gamma (r, λ), $M_X(t) = (\frac{\lambda}{\lambda - t})^r$, $t < \lambda$.

let $S = \sum_{i=1}^{100} X_i$

$$M_S(t) = M_{X_1}(t) \cdots M_{X_{100}}(t) = \left(\left(\frac{.02}{.02-t}\right)^{.01}\right)^{100} = \left(\frac{.02}{.02-t}\right)^{1}$$

$$M_Y(t) = M_{\frac{1}{100}S}(t) = M_S(.01\,t) = \frac{.02}{.02-.01t} = \frac{2}{2-t}$$

By the uniqueness of MGF, $Y \sim \text{Exp}(2)$

② **Recognizing a distribution from the variable part of its density.**

A density can be written as

$$f(t) = c\, h(t)$$

c: constant, $h(t)$: variable part.

$$1 = \int_{-\infty}^{\infty} f(t)\,dt = c \int_{-\infty}^{\infty} h(t)\,dt \quad \Rightarrow \quad c = \frac{1}{\int_{-\infty}^{\infty} h(t)\,dt}$$

So you can figure out the density from its variable part.

List of densities. Please circle their variable parts:

ex: $T \sim \text{Gamma}(r, \lambda)$ $\quad f(t) = \frac{1}{\Gamma(r)} \lambda^r\, t^{r-1} e^{-\lambda t}, \; t > 0$

$T \sim \text{Normal}(\mu, \sigma^2)$ $\quad f(t) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$

$T \sim \text{Unif}(a, b)$ $\quad f(t) = \frac{1}{b-a}\, \mathbb{1}_{(t \in (a,b))}$

$T_r \sim \text{Gamma}(r, \lambda)$, $r, \lambda > 0$ $\quad f(t) = \begin{cases} \frac{1}{\Gamma(r)} \lambda^r t^{r-1} e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$

$T \sim \text{Exp}(\lambda)$, $\lambda > 0$ $\quad f(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & \text{else} \end{cases}$ ← variable part

ex Name the distribution with the following variable part ex $\text{Gamma}(r = \frac{1}{2}, \lambda = 3)$

a) $h(t) = t^3 e^{-\frac{1}{2}t}$ Gamma$(4, \frac{1}{2})$

b) $h(t) = e^{-\frac{1}{2}t^2}$ Normal$(0,1)$

c) $h(t) = e^{-3t}$ Exp(3)

d) $h(t) = t^{-\frac{1}{2}} e^{-t}$ Gamma$(\frac{1}{2}, 1)$

e) $h(t) = \mathbb{1}_{(t \in (0,1))}$ Unif$(0,1)$

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Let X be the standard normal RV. The distribution of $Y = X^2$ is:

a Gamma$(\frac{1}{2}, \frac{1}{2})$

b Gamma$(\frac{3}{2}, \frac{1}{2})$

c Normal(0,1)

d none of the above

$g(x) = x^2$

$g'(x) = 2x$

$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$f_Y(y) = \sum_{\{x \mid g(x)=y\}} \frac{f_X(x)}{|g'(x)|}\Bigg|_{x=g^{-1}(y)}$$

evaluated at $x=g^{-1}(y)$

$$f(y) = \frac{f(x)}{|2x|}\Bigg|_{x=\pm\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{e^{-y/2}}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}}\frac{e^{-y/2}}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-1/2\, y}$$

variable part of Gamma$(\frac{1}{2}, \frac{1}{2})$