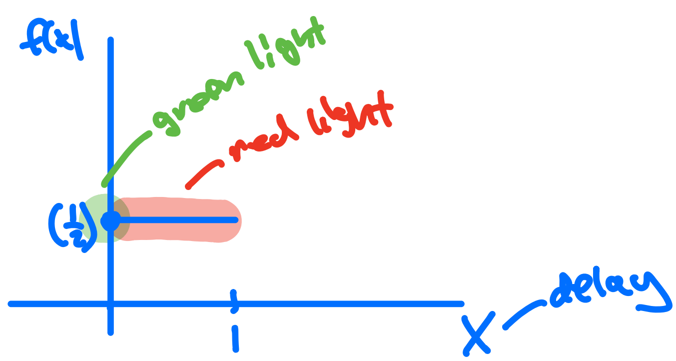
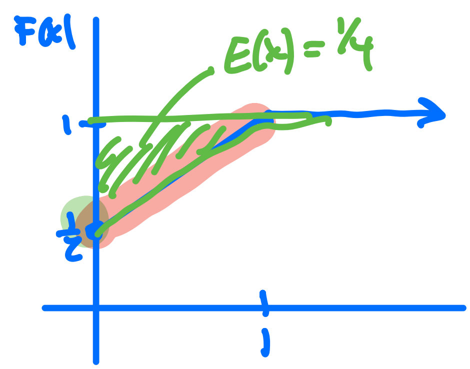
Stat 134 lec 27

warmup: 11:00-11:10

Suppose stop lights at an intersection alternately show green for one minute, and red for one minute (no yellow). Suppose a car arrives at the lights at a time distributed uniformly from 0 to 2 minutes. Let X be the delay of the car at the lights (assuming there is only one car on the road). Graph the density and the cdf of X. Also find E(X)





Last time

sec 4.5 Expectation of a nonnegative RV using CDF

$$E(X) = \int_0^\infty (1 - F(x))\,dx$$

ex let $X \sim \text{Geom}\left(\frac{1}{2}\right)$

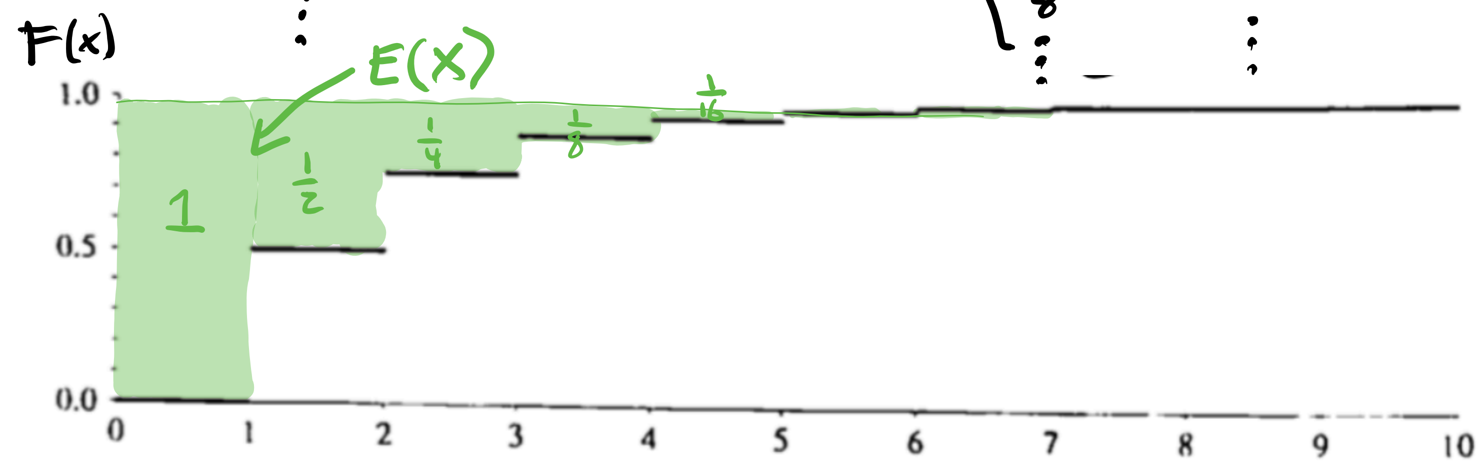
$P(X=1) = 1/2$

$P(X=2) = \frac{1}{2}\cdot\frac{1}{2} = 1/4$

Picture $P(X=3) = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} = 1/8$

$\vdots$

$$\Rightarrow F(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ \frac{7}{8} & 3 \le x < 4 \\ \vdots & \vdots \end{cases}$$



$$E(X) = \int_0^\infty (1 - F(x))\,dx$$

$$E(X) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

p

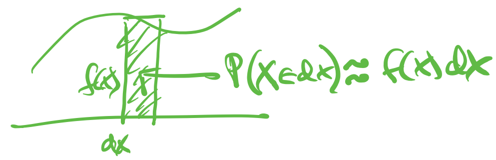
$$= \sum_{j=0}^{\infty} \underset{q^j}{P(X > j)} = \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \quad \leftarrow \text{tail sum formula.}$$

Today

① Overview of what we have learned since the midterm.

② sec 4.6 Order statistics

① Overview



Chap 4

density of distribution

single variable unconditional prob

- change of variable formula for densities,
- expectation
- continuous distributions
  - uniform
  - exponential/gamma
  - order statistics / beta
- MGF – useful tool
  - calculate moments
  - identify a distribution by its MGF
- CDF / mixed distributions

calculating expectation from cdf.

Chap 5

multiple variable unconditional prob

- joint distributions

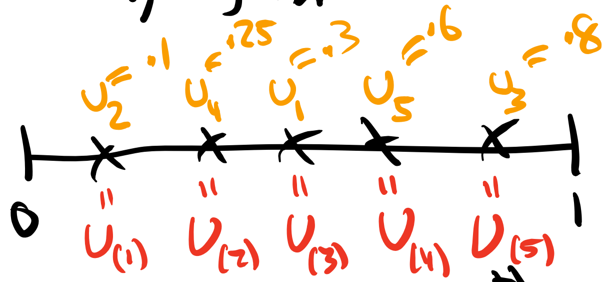
Chap 6

multiple variable conditional prob.

- dependence

① Sec 4.6 order statistic of $U(0,1)$

let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$



throw 5 darts at $[0,1]$.

let $U_{(k)}$ = called the k^{th} order statistic

= k^{th} largest value of $U_1, \dots, U_n$

(assuming no ties)

ex

$U_{(1)} = \min(U_1, \dots, U_n)$

$U_{(n)} = \max(U_1, \dots, U_n)$

Review counting

You have 3 red, 2 green and 5 blue marbles, How many orderings of these 10 marbles are there?

ex rrr gg bbbbb

grrr g bbbbb

ggrrr bbbbb

⋮

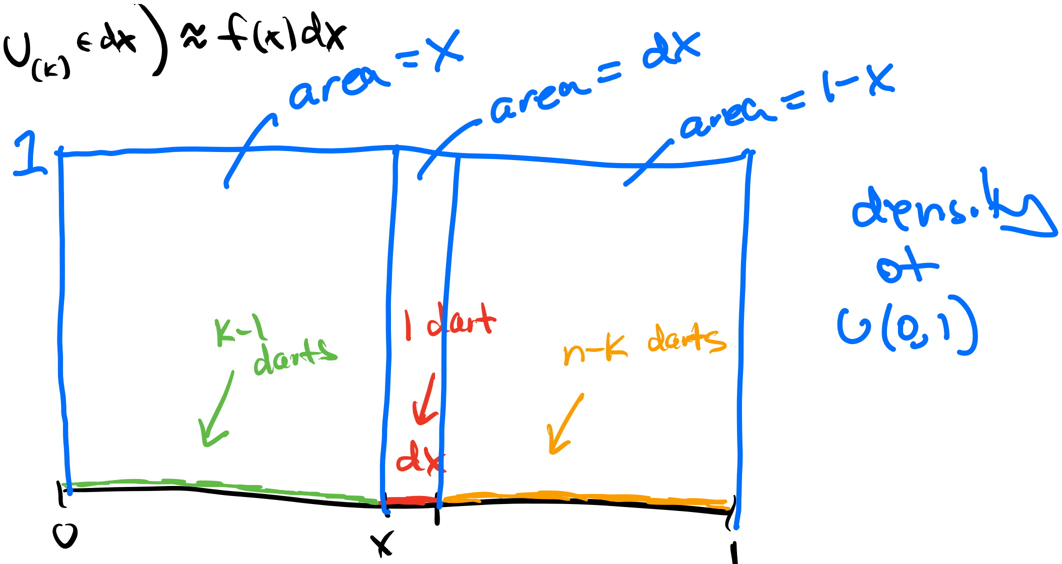
$$\binom{10}{3,2,5} = \binom{10}{3} \cdot \binom{7}{2} \binom{5}{5}$$

$$= \frac{10!}{3!\,2!\,5!}$$

Next, find density of $U_{(k)}$

write $P(U_{(k)} \in dx) \approx f(x)\,dx$

Picture



$U_{(k)} \in dx$ means that k-1 darts are between 0 and x, and one is in dx, and n-k darts are between x and 1

$$P(U_{(k)} \in dx) = P\Big(k\text{-}1 \text{ darts} \in (0,x),\ 1 \text{ dart} \in dx,\ n\text{-}k \text{ darts} \in (x,1)\Big)$$

$$= P\big(k\text{-}1 \text{ darts} \in (0,x)\big) \cdot P\big(1 \text{ dart} \in dx \,\big|\, k\text{-}1 \text{ darts} \in (0,x)\big)$$

$$\cdot P\big(n\text{-}k \text{ darts} \in (x,1) \,\big|\, 1 \text{ dart} \in dx,\ k\text{-}1 \text{ darts} \in (0,x)\big)$$

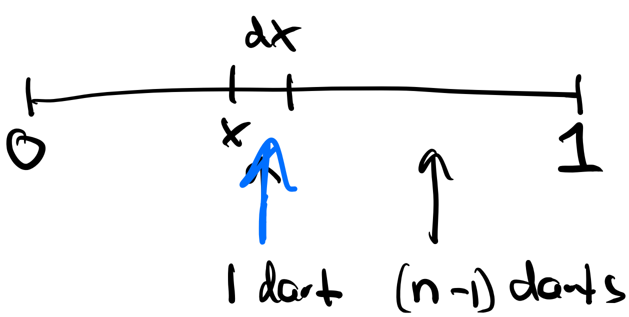
$$= \binom{n}{k-1} x^{k-1} \binom{n-k+1}{1} 1\,dx \binom{n-k}{n-k} (1-x)^{n-k}$$

$$= \underbrace{\binom{n}{k-1,\,1,\,n-k} x^{k-1} (1-x)^{(n-k+1)-1}}_{f_{U_{(k)}}(x)} \, dx$$

$$\Rightarrow f_{U_{(k)}}(x) = \binom{n}{k-1,\ 1,\ n-k} x^{k-1} (1-x)^{(n-k+1)-1} \quad \text{for } 0<x<1$$

Ex Let $U_1, \ldots, U_n \overset{iid}{\sim} \mathrm{Unif}(0,1)$

Find the density of $U_{(1)}$

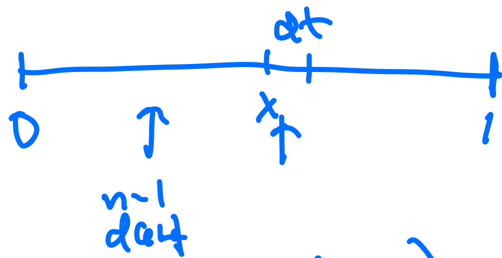


$$P(X_{(1)} \in dx) = \binom{n}{1,\ n-1} dx\, (1-x)^{n-1} = n(1-x)^{n-1} dx$$

$$\Rightarrow \boxed{f_{U_{(1)}}(x) = n(1-x)^{n-1}}$$

Ex Let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$

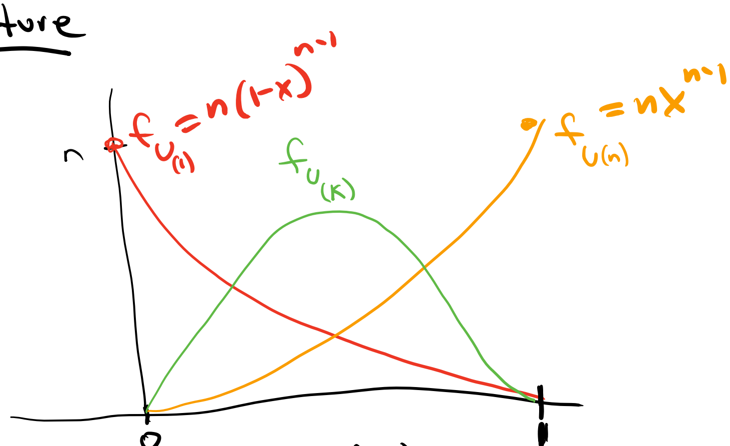
Find the density of $U_{(n)}$



$$f_{U_{(n)}}(x)\,dx = \binom{n}{n-1,\,1} x^{n-1} \cdot dx$$

$$\Rightarrow f_{U_{(n)}}(x) = n x^{n-1}$$

Picture

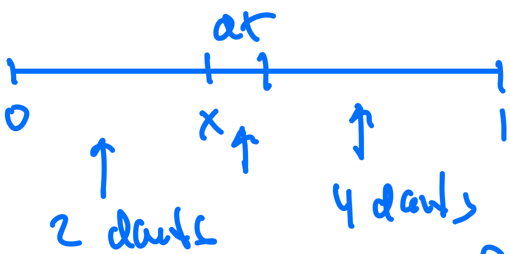


Order statistic of $U(0,1)$ provides a family of densities on the unit interval.

ex

$x^2(1-x)^4$ for $0<x<1$ is the variable part of density of what RV? How many darts do you throw?

$U_{(3)}$ of $n=7$



$P(U_{(3)} \in dx) = f(x)dx$

$$f_{U_{(3)}}(x) = \binom{7}{2,1,4} x^2(1-x)^4$$