Stat 134 lec 28

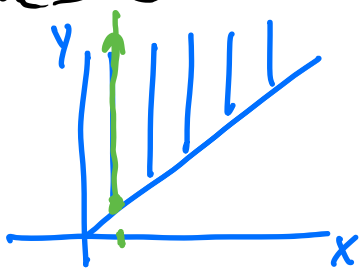
warmup 9:00-9:10

Let $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$

(recall, $f_X(x) = \lambda e^{-\lambda x}$)

be independent lifetimes of two bulbs.

Find $P(X<Y)$.



Hint: use $f(x,y) = f_X(x) f_Y(y)$

$$f(x,y) = \lambda e^{-\lambda x} \mu e^{-\mu y}$$

$$P(X<Y) = \lambda \cancel{\mu} \int_{x=0}^{x=\infty} e^{-\lambda x}\, dx \underbrace{\int_{y=x}^{y=\infty} e^{-\mu y}\, dy}_{= \frac{e^{-\lambda x}}{\cancel{\mu}}}$$

$$= \lambda \int_{x=0}^{x=\infty} e^{-(\mu+\lambda)x}\, dx = \boxed{\frac{\lambda}{\mu+\lambda}}$$

dxdy

dydx

Last time.

Sec 4.6 Beta Distribution

Let $r, s > 0$

$P \sim \text{Beta}(r,s)$ if

$$f(p) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1} \quad \text{for } 0<p<1$$

$$E(X) = \frac{r}{r+s}$$

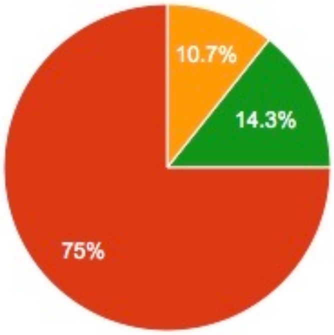
Applications

a) Beta(r, s) takes values between 0 and 1 and commonly models the prior distribution of a probability in Bayesian statistics.

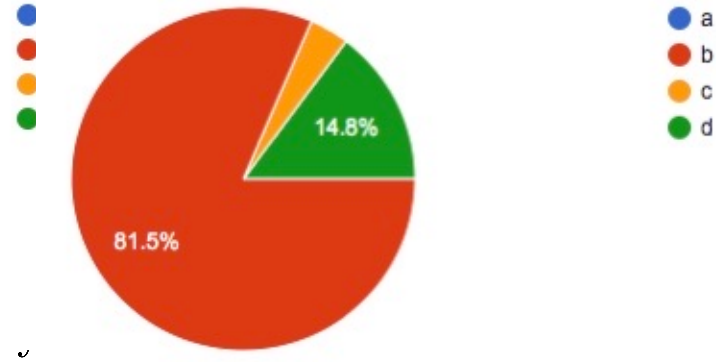
b) generalization of standard uniform ordered statistic

If throw n darts at $[0,1]$

$U_{(k)} \sim \text{Beta}(k, n-k+1)$



⇒



1. Let P be the chance a coin lands head. Suppose the prior distribution of P is $f_P(p) = c(1-p)^4$ for $0 \le p \le 1$ for some constant c. Which of the following is true:

a $P \sim Beta(1, 4)$

b $c = 5$

c $E(P) = \frac{1}{5}$

d more than one of the above

b	For there to be no x term and for 1-x to be of power 4, r must be 1 and s must be 5. Therefore b is the only option.

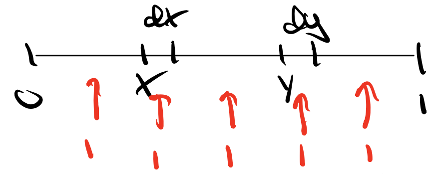
Sec <u>5.1, 5.2</u> Joint Density

Throw down 5 darts on (0,1).

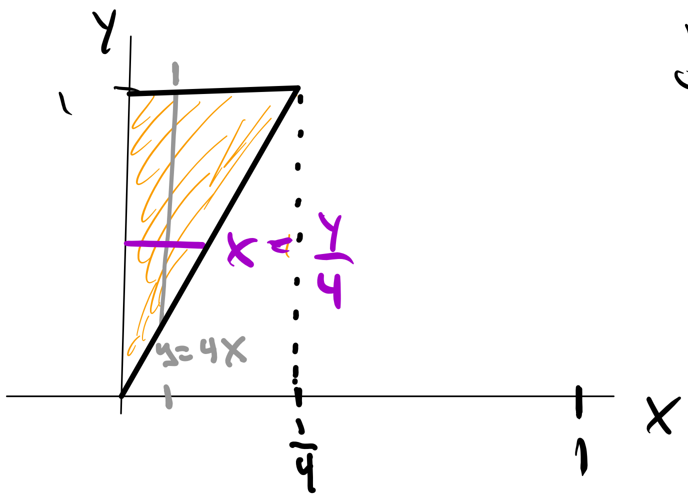
ex $X = U_{(2)}$ $Y = U_{(4)}$

Find $P(Y > 4X)$

recall,



$$f(x,y) = \binom{5}{1,1,1,1,1} x(y-x)(1-y)$$
$$= \boxed{5!\, x(y-x)(1-y)}$$



What are bounds of integrals?

$$P(Y>4X) = \int_{y=0}^{y=1} \int_{x=0}^{x=\frac{y}{4}} 5!\, x(y-x)(1-y)\,dx\,dy$$

(or)

$$P(Y>4X) = \int_{x=0}^{x=1/4} \int_{y=4x}^{y=1} 5!\, x(y-x)(1-y)\,dy\,dx$$

see appendix for solution of double integral.

Today

1. Sec 5.1, 5.2 Independent RVs
2. Sec 5.2 Competing exponentials
3. Sec 5.2 Marginal density

① Sec 5.1, 5.2

Independent RVs

defn X and Y are independent if

$$P(X\in dx, Y\in dy) = P(X\in dx)\, P(Y\in dy)$$

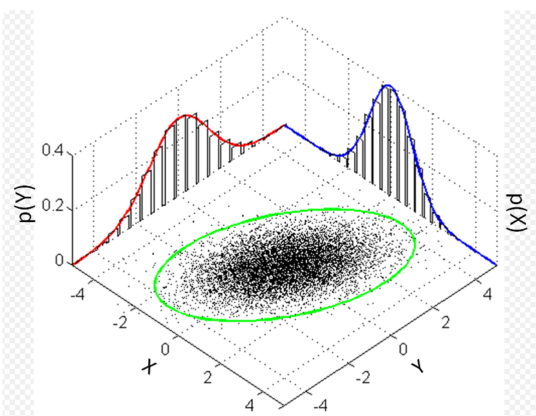
$$\underset{f(x,y)\,dx\,dy}{\shortparallel} \qquad \underset{f(x)\,dx}{\shortparallel} \qquad \underset{f(y)\,dy}{\shortparallel}$$

$$\Longleftrightarrow \boxed{f(x,y) = f(x)f(y)}$$

ex $X, Y \overset{iid}{\sim} N(0,1)$

$$f(x,y) = \phi(x)\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$



Not a great picture because the oval in green should be a circle. This is the picture of a correlated bivariate normal from chapter 6 instead of an uncorrelated bivariate normal.

ex If $X, Y \overset{iid}{\sim} U(0,1)$

Find $P(Y \geq \frac{1}{2} \mid Y \geq 1-2X)$

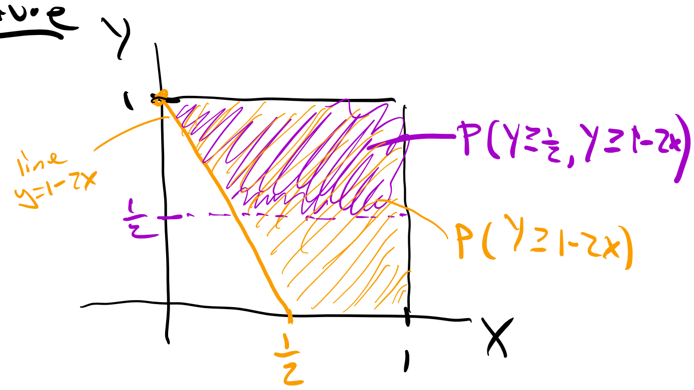
Soln

$f(x,y) = f(x)f(y) = 1$ for $0 < x, y < 1$, 0 else.

(with $f(x) = 1$ and $f(y) = 1$)

$$P(Y \geq \tfrac{1}{2} \mid Y \geq 1-2X) = \frac{P(Y \geq \frac{1}{2}, Y \geq 1-2X)}{P(Y \geq 1-2X)}$$ Bayes' rule

Picture



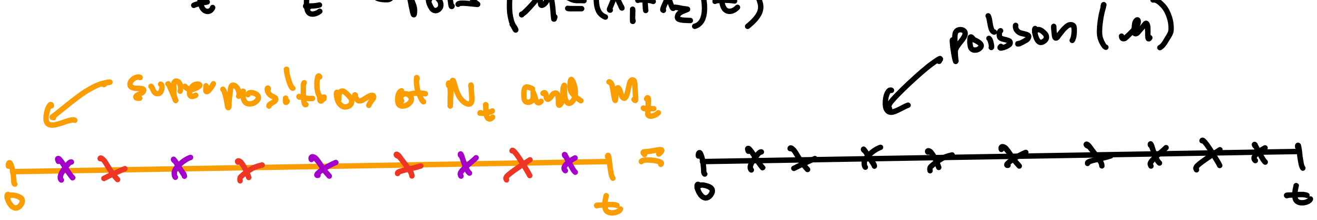
$$\frac{\frac{1}{2} - \frac{1}{16} = \frac{7}{16}}{1 - \frac{1}{4} = \frac{3}{4}} = \boxed{\frac{7}{12}}$$

② sec 5.2 Competing exponentials

superposition of Poisson random scatters:

let $N_t \sim Pois(\mu_1 = \lambda_1 t)$ and $M_t \sim Pois(\mu_2 = \lambda_2 t)$ be independent PRS corresponding to the number of arrivals of red and purple cars in time t.

Then $N_t + M_t \sim Pois(\mu = (\lambda_1 + \lambda_2)t)$



Competing exponentials:

Let X = time until the first red car

Y = time until the first purple car

What is the chance the first car is red?

$$P(X<Y) = \frac{\lambda_1}{\lambda_1+\lambda_2}$$

from warmup.

= Prob second car is red = prob a car is red.

color of first car independent of color of 2nd car.

By Poisson thinning

$$N_t \sim Pois\left(p\cdot(\mu_1+\mu_2)\right) = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}(\lambda_1+\lambda_2)t\right) = \mu_1 = \lambda_1 t \checkmark$$

where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$

and $\mu_1+\mu_2 = (\lambda_1+\lambda_2)t$

ex

(Exponential distributions) You are first in line to have your question answered by either of the 2 uGSIs, Brian and Yiming, who are busy with their current students. Your question will be answered as soon as the first uGSI finishes. The times spent with student by Brian and Yiming are independent and exponentially distributed with (positive) rates λ_B and λ_Y respectively, i.e. Brian's distribution is Exponential(λ_B), and Yiming's is Exponential(λ_Y).

(a) Find the probability that Yiming will be the one answering your questions.

$$P(Y < B) = \frac{\lambda_Y}{\lambda_Y + \lambda_B}$$

(b) What is the distribution of your wait time? Your answer should not include integrals.

$$W = \min(B, Y) \sim \text{Exp}(\lambda_Y + \lambda_B)$$

$$P(W > w) = P(\min(B, Y) > w)$$
$$= P(Y > w, B > w) = P(Y > w) P(B > w)$$
$$= e^{-\lambda_Y w} \cdot e^{-\lambda_B w}$$
$$= e^{-(\lambda_Y + \lambda_B) w}$$

Think of min(Y,B) as a superposition of Poisson processes with rate $\lambda_Y + \lambda_B$.

tinyurl.com/oct28-2022

Stat 134

Friday October 21 2022

1. You are first in line to have your question answered by either of the 3 uGSI Yiming, Brian and Rowan, whose wait time to be seen, Y, B and R, are independent and exponentially distributed RVs with rates λ_Y, λ_B, and λ_R respectively. $P(Y < B < R)$ is?

   a $\frac{\lambda_Y+\lambda_B}{\lambda_Y+\lambda_B+\lambda_R}$

   b $\frac{\lambda_Y}{\lambda_Y+\lambda_B+\lambda_R} \times \frac{\lambda_B}{\lambda_B+\lambda_R}$

   c $\frac{\lambda_Y}{\lambda_Y+\lambda_B} \times \frac{\lambda_B}{\lambda_B+\lambda_R}$

   d none of the above

$$P\left(Y=\min(Y,B,R),\ B=\min(B,R)\right)$$

$$= P\left(Y=\min(Y,B,R)\right) \cdot P\left(B=\min(B,R) \mid Y=\min(Y,B,R)\right)$$

$$\boxed{\frac{\lambda_Y}{\lambda_Y+\lambda_B+\lambda_R} \cdot \frac{\lambda_B}{\lambda_B+\lambda_R}}$$

Appendix

details:

$$P(Y > 4x) = \int_{y=0}^{y=1} \int_{x=0}^{y/4} 120x(y-x)(1-y)\,dx\,dy$$

$$= \int_{y=0}^{y=1} 120(1-y) \int_{x=0}^{x=y/4} (xy - x^2)\,dx\,dy$$

$$= \int_{y=0}^{y=1} 120(1-y) \left[\frac{x^2 y}{2} - \frac{x^3}{3} \right] \Bigg|_{x=0}^{x=y/4} dy$$

$$= \int_{y=0}^{y=1} 120(1-y) \underbrace{\left(\frac{y^3}{32} - \frac{y^3}{3 \cdot 64} \right)}_{\frac{6y^3 - y^3}{3 \cdot 64} = \frac{5y^3}{192}} dy$$

$$= \frac{5}{192} \cdot 120 \int_{y=0}^{y=1} y^3 - y^4\,dy$$

$$= \frac{5}{192} \cdot 120 \int_0^1 y^3 - y^4\,dy = \frac{5 \cdot 120}{192} \left(\frac{y^4}{4} - \frac{y^5}{5} \right) \Bigg|_0^1$$

$$= \frac{5 \cdot 120}{192} \left(\frac{1}{4} - \frac{1}{5} \right) = \frac{30}{192} = .156$$