Stat 134 lec 29

warmup 9:00-9:10

ex (5.2.9a)

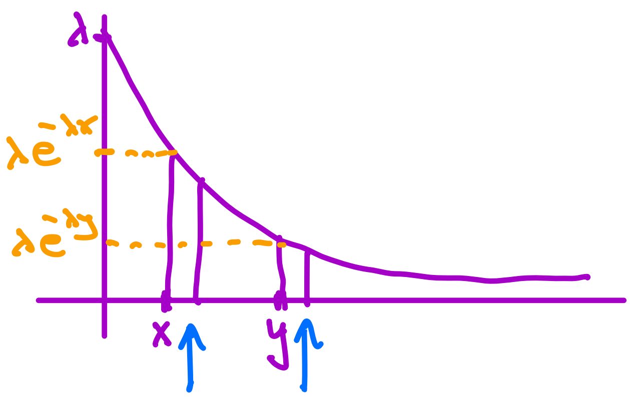
$S, T \overset{iid}{\sim} Exp(\lambda)$ $\left(f_S(s) = \lambda e^{-\lambda s} \right)$

$X = \min(S,T)$ ← 1^{st} ordered statistic of Exp(λ)

$Y = \max(S,T)$ ← 2^{nd} ordered statistic of Exp(λ)

Find the joint density of X and Y

Picture



idea $P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$

$$= \binom{2}{1} \lambda e^{-\lambda x} dx \binom{1}{1} \lambda e^{-\lambda y} dy = \underbrace{2\lambda^2 e^{-\lambda(x+y)}}_{= f(x,y)} dx\,dy$$

$$\boxed{f(x,y) = 2\lambda^2 e^{-\lambda(x+y)} \quad \text{for } 0 < x < y}$$

Earlier material

$T \sim Exp(\lambda)$, $cT \sim Exp\left(\frac{\lambda}{c}\right)$

Competing exponentials

$T_1 \sim Exp(\lambda_1)$, $T_2 \sim Exp(\lambda_2)$ $\Rightarrow P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

Properties of std normal Z	Proved it
$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$	no
$E(Z) = 0$ — Even though Z is symmetric around zero it is possible $E(Z)$ is undefined; ex Cauchy distribution	no
$SD(Z) = 1$	no

Let $X, Y \overset{iid}{\sim} N(0,1)$ with density $\phi(x) = ce^{-\frac{1}{2}x^2}$, $c > 0$

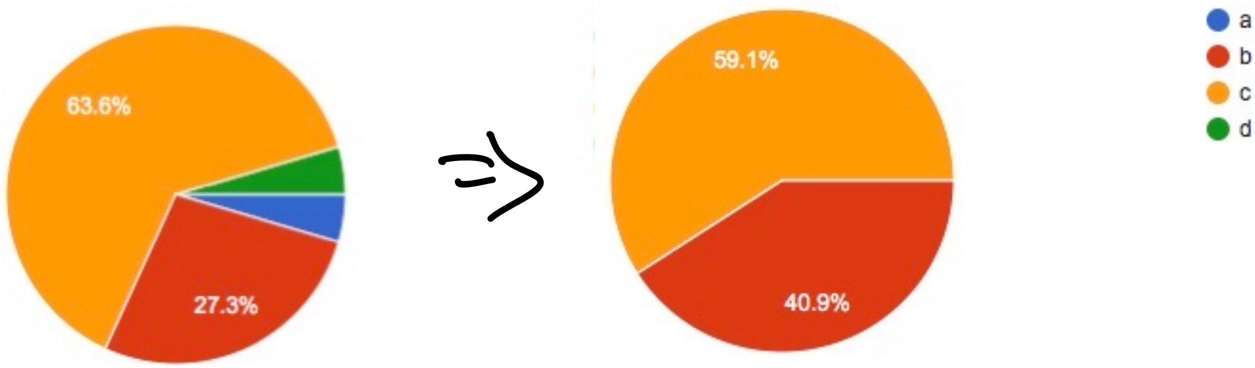
$f(x,y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ for $c > 0$

We still need to show that $c = \frac{1}{\sqrt{2\pi}}$, $E(X) = 0$, $SD(X) = 1$.

Sec 5.2 Competing exponentials

If $X_1, \ldots, X_n$ are indep exponentials with rates $\lambda_1, \ldots, \lambda_n$

$$P(X_i = \min(X_1, \ldots, X_n)) = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$$



1. You are first in line to have your question answered by either of the 3 uGSI Yiming, Brian and Rowan, whose wait time to be seen, Y, B and R, are independent and exponentially distributed RVs with rates λ_Y, λ_B, and λ_R respectively. $P(Y < B < R)$ is?

   a $\frac{\lambda_Y+\lambda_B}{\lambda_Y+\lambda_B+\lambda_R}$

   b $\frac{\lambda_Y}{\lambda_Y+\lambda_B+\lambda_R} \times \frac{\lambda_B}{\lambda_B+\lambda_R}$

   c $\frac{\lambda_Y}{\lambda_Y+\lambda_B} \times \frac{\lambda_B}{\lambda_B+\lambda_R}$

   d none of the above

c	P(Y<B<R) = P(Y<B)*P(B<R)

Y<B and B<R not independent

if Rowen is super fast less likely that Y<B

$P(Y<B \mid B<R) \leq P(Y<B)$

$$P(Y<B<R) = P(Y<B, B<R) = P(Y<B \mid B<R)\,P(B<R)$$

$$P(Y<B \mid B<R) = P\big(Y = \min(Y,B,R)\big) = \frac{\lambda_Y}{\lambda_Y+\lambda_B+\lambda_R}$$

$$\Rightarrow P(Y<B<R) = \frac{\lambda_Y}{\lambda_Y+\lambda_B+\lambda_R} \cdot \frac{\lambda_B}{\lambda_B+\lambda_R}$$

b	I did the integral

$$P(Y<B<R) = \int_{y=0}^{\infty} \lambda_Y e^{-\lambda_Y y} \int_{B=y}^{\infty} \lambda_B e^{-\lambda_B b} \int_{R=b}^{\infty} \lambda_R e^{-\lambda_R r}\, dr\, db\, dy$$

$$\int_{R=b}^{\infty} \lambda_R e^{-\lambda_R r}\, dr = e^{-\lambda_R b}$$

$$= \lambda_Y \lambda_B \int_{y=0}^{\infty} e^{-\lambda_Y y} \int_{B=y}^{\infty} e^{-(\lambda_B+\lambda_R) b}\, db\, dy \quad \text{etc}$$

$$\int_{B=y}^{\infty} e^{-(\lambda_B+\lambda_R) b}\, db = \frac{e^{-(\lambda_B+\lambda_R) y}}{\lambda_B+\lambda_R}$$

etc ...

Today

1. Sec 5.2 Marginal Densities
2. Sec 5.2 Expectation $E(g(X,Y))$
3. Sec 5.3 Rayleigh distribution

① Sec 5.2 Marginal densities

Recall marginal probability:

discrete picture

Y \ X	0	1	$P(y)$
2	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$
$P(x)$	$\frac{1}{2}$	$\frac{1}{2}$	

marginal probability of X

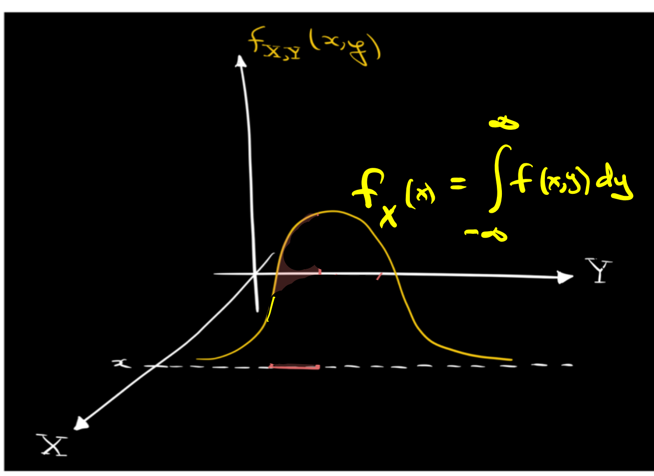
$$P(x) = \sum_{y \in Y} P(x,y)$$

Marginal prob of Y

$$P(y) = \sum_{x \in X} P(x,y)$$

2

Continuous picture: marginal density



ex S and T are iid Exp(λ)

$X = \min(S,T)$ and $Y = \max(S,T)$.

The joint density is

$f(x,y) = 2\lambda^2 e^{-\lambda(x+y)}$ for $0<x<y$,

0 else.

Find the marginal of Y.

$$f_Y(y) = \int_{x=-\infty}^{x=\infty} f(x,y)\,dx$$

$$= 2\lambda^2 e^{-\lambda y} \underbrace{\int_{x=0}^{x=y} e^{-\lambda x}\,dx}_{\frac{1-e^{-\lambda y}}{\lambda}}$$

$$= \boxed{2\lambda e^{-\lambda y}\left(1-e^{-\lambda y}\right) \quad y>0}$$

② <u>Sec 5.2</u> Expectation $E(g(X,Y))$

Let (X,Y) have joint density $f(x,y)$ and $g(X,Y)$ be a function of X,Y.

Define

$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y)f(x,y)\,dx\,dy$$

ex

joint density

$$f(x,y) = \begin{cases} 30\,(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$

Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\,dx\,dy$$

$g(x,y) = Y$

we know $Y = U_{(6)} \sim \text{Beta}(6,1) \Rightarrow E(Y) = \frac{6}{6+1} = \boxed{\frac{6}{7}}$

$k \quad n-k+1 = 6-6+1 = 1$

See appendix to notes.

③ Sec 5.3 Rayleigh Distribution

let $T \sim Exp(\frac{1}{2})$, $f(t) = \frac{1}{2}e^{-\frac{1}{2}t}$, $t > 0$

$R = \sqrt{T}$ ← R is called the Rayleigh Distribution

write $R \sim Ray$

$$P(R > r) = P(R^2 > r^2) = P(T > r^2)$$

$$= e^{-\frac{1}{2}r^2}$$

Exp($\frac{1}{2}$)

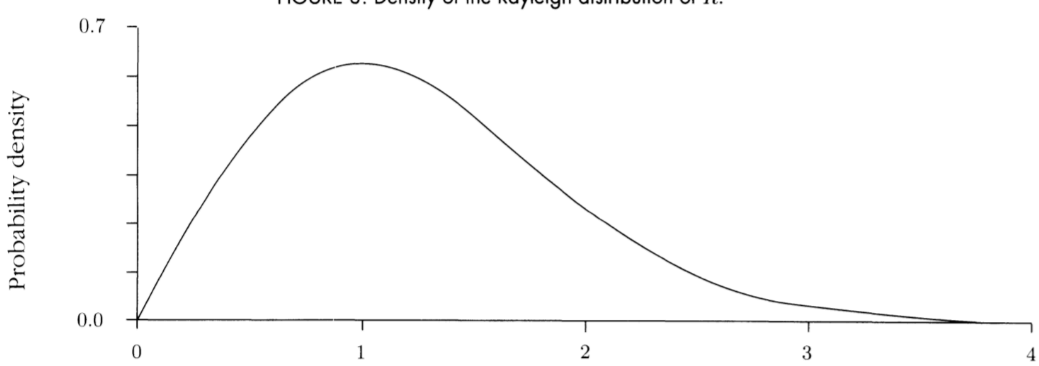
So $\boxed{F_R(r) = 1 - e^{-\frac{1}{2}r^2}, \quad r > 0}$

$$f_R(r) = \frac{d}{dr}F_R(r)$$

$$= \frac{1}{2}e^{-\frac{1}{2}r^2} \cdot 2r = \boxed{re^{-\frac{1}{2}r^2}, \quad r > 0}$$

so $\boxed{\int_{r=0}^{r=\infty} re^{-\frac{1}{2}r^2}\,dr = 1}$

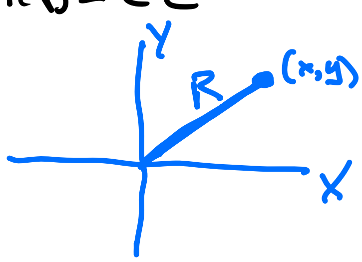
FIGURE 3. Density of the Rayleigh distribution of R.



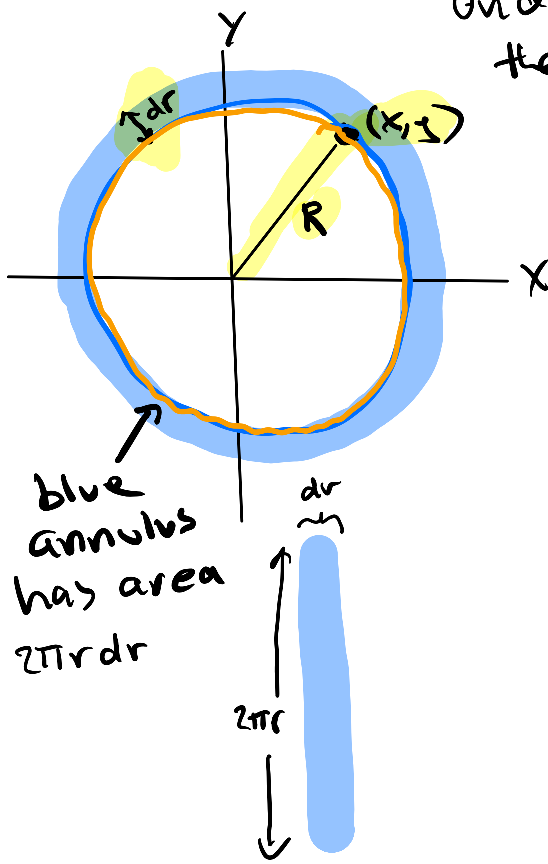
An interesting calculation:

For $X, Y \overset{iid}{\sim} N(0,1)$, $f(x,y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ for $c > 0$

Let $R = \sqrt{X^2 + Y^2}$



$P(R \in dr)$ is the volume of the cylinder under $f(x,y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$ over the shaded blue annulus.



$P(R \in dr) \approx$ height of $f(x,y)$ above blue annulus

$\cdot$ area of blue annulus

$\quad\; = f_R(r)\,dr$

$$= c^2 e^{-\frac{1}{2}r^2} \cdot (2\pi r\,dr)$$

$$= c^2 2\pi \, \underbrace{r e^{-\frac{1}{2}r^2}}_{\text{Rayleigh Density}} dr$$

$$1 = \int_{r=0}^{r=\infty} \underbrace{P(R \in dr)}_{= f_R(r)\,dr} = c^2 2\pi \underbrace{\int_0^\infty r e^{-\frac{1}{2}r^2} dr}_{=1}$$

$$\Rightarrow 1 = c^2 2\pi \rightarrow c = \frac{1}{\sqrt{2\pi}}$$

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Stat 134

Friday October 21 2022

1. From the "interesting calculation" involving X, Y iid $N(0,1)$ you can conclude:

a $f_X(x) = ce^{-x^2/2} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is a density

b $\sqrt{X^2+Y^2}$ is $Exp(\frac{1}{2})$

c $\sqrt{X^2+Y^2}$ is Rayleigh

d more than one of the above

– see end of lecture notes

Also, $E(X)=0$ and $SD(X)=1$

Appendix

Expectation $E(g(X,Y))$

$$E(g(x,y)) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} g(x,y) f(x,y)\,dx\,dy$$

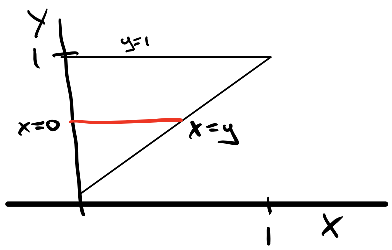
ex

joint density

$$f(x,y) = \begin{cases} 30(y-x)^4 & 0<x<y<1 \\ 0 & \text{else} \end{cases}$$

$X = U_{(1)}$

$Y = U_{(6)}$



Find

$$E(Y) = \int_{y=-\infty}^{y=\infty}\int_{x=-\infty}^{x=\infty} y\, f(x,y)\,dx\,dy$$

$g(x,y) = Y$

$$= \int_{y=0}^{y=1} y \int_{x=-\infty}^{x=\infty} f(x,y)\,dx\,dy$$

showed this is lecture 29

$f_Y(y) = 6y^5$

$$= 6\int_{y=0}^{y=1} y^6\,dy = \frac{6y^7}{7}\Big|_0^1 = \boxed{6/7}$$

Appendix

Claim Let $X \sim N(0,1)$. We show that $E(X)=0$ and $SD(X)=1$.

Let $X \sim N(0,1)$

Show $E(X)=0$

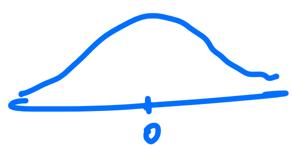
$\phi(x)$ is symmetric about zero so all we have to show is that $E(X)$ converges absolutely.

(i.e $E(|X|) < \infty$).

$$E(X) = \int x \, \underset{\phi(x)}{f(x)} \, dx$$

$$E(|X|) = \int_{-\infty}^{\infty} |x| \phi(x) dx$$

$$= 2 \int_0^{\infty} x \phi(x) dx$$



$$= 2 \int_0^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$E(|X|) = 2\int_0^{\infty} x f(x) dx < \infty \quad \checkmark$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \underbrace{\int_0^{\infty} x e^{-\frac{x^2}{2}} dx}_{=1} $$

Rayleigh density

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \underset{\infty}{\int_{-\infty}^{0} x f(x) dx} + \underset{\infty}{\int_0^{\infty} x f(x) dx}$$

$$= \sqrt{\frac{2}{\pi}} < \infty.$$

$\Rightarrow E(X)=0$ since $\phi(x)$ is semetric around zero

Next we show $SD(X) = 1$:

We know $X^2 + Y^2 \sim Exp\left(\frac{1}{2}\right)$ from above

$\frac{1}{2} = \lambda$

so $E(X^2+Y^2) = \frac{1}{\frac{1}{2}} = 2$

λ rate of X^2+Y^2

$$\Rightarrow \underbrace{E(X^2)}_{} + \overset{E(X^2)}{\overset{\parallel}{E(Y^2)}} = 2$$

$\Rightarrow 2E(X^2) = 2 \Rightarrow E(X^2) = 1$

but $SD(X) = \sqrt{E(X^2) - E(X)^2}$

$= \sqrt{1-0} = 1$ ✓