Stat 134 lec 31

Warmup 9:00–9:10

Let $X, Y \overset{iid}{\sim} N(0,1)$

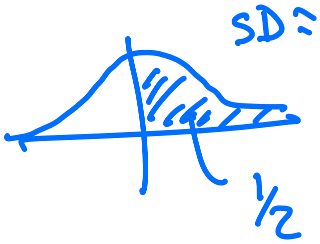
Find $P(X > 2Y)$

$\parallel$

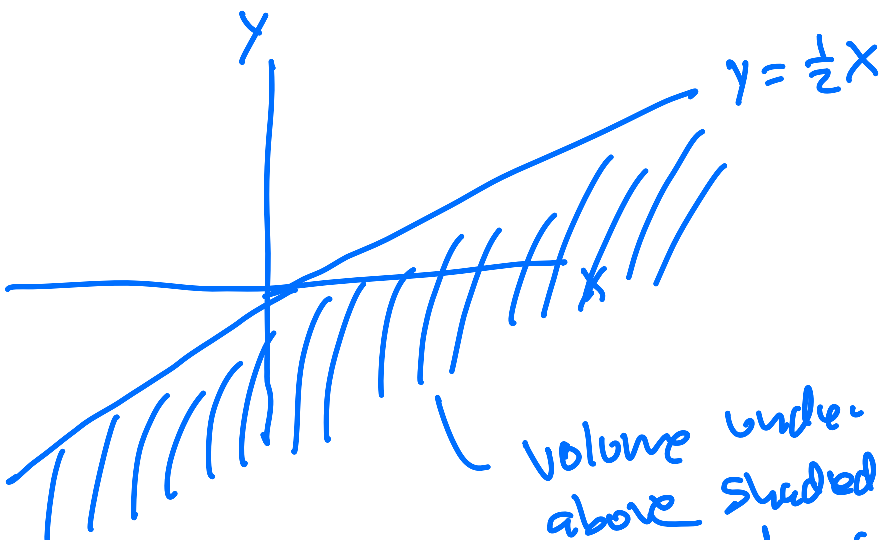
$P(X - 2Y > 0)$

$X - 2Y \sim N(0, 5)$

$SD = \sqrt{5}$



$P(X - 2Y > 0) = 1/2$



volume under joint $f(x,y)$ above shaded region is $1/2$ by symmetry of bell shaped density.

Last time

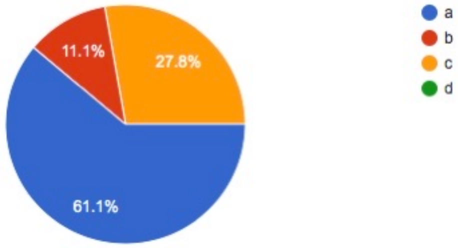
Sec 5.3

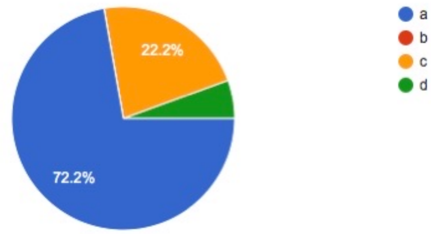
A linear combination of independent normals is normal.

Thm Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$ indep.

then $aX_1 + bX_2 \sim N(a\mu_1 + b\mu_2,\ a^2\sigma_1^2 + b^2\sigma_2^2)$

Note In chapter 6 we will generalize this result and show that $aX_1 + bX_2$ is normal iff (X_1, X_2) are bivariate normal





2. Let $X \sim N(68, 3^2)$ and $Y \sim N(66, 2^2)$ be independent. $P(X > Y)$ equals

a $1 - \Phi(\frac{0-2}{\sqrt{3^2+2^2}})$

b $1 - \Phi(\frac{0-2}{3^2+2^2})$

c $1 - \Phi(\frac{68-66}{\sqrt{3^2+2^2}})$

d none of the above

c	P(X>Y)=P(X-Y>0)=1-P(X-Y<0)=c

a	X-Y~N(68-66, 9+4)=N(2, 13) P(X-Y>0) = 1-P(X-Y<0) = 1-phi(-2/sqrt(13))——>A

Today

Sec 5.4

① Convolution formula for sum

② Triangular density

③ other convolution formulas

(1) General convolution formula

We can write a general convolution formula for any operation.

1 dimensional change of variables

RV	transformed RV
Y	$Z(Y)$ a differentiable function
f_Y	$f_Z = \left\lvert \frac{\partial y}{\partial z} \right\rvert f_Y$

$\frac{1}{\left\lvert \frac{\partial z}{\partial y} \right\rvert}$

ex $Z = Y^3$

2 dimensional change of variables

RV	transformed RV
(X, Y)	$(X, Z(X,Y))$ a differentiable function
$f_{X,Y}$	$f_{X,Z}$

ex $Z = X + Y$

ex $Z = \frac{X}{Y}$, $Y \neq 0$

$$f_{X,Z} = \left\lvert \det \frac{\partial(X,Y)}{\partial(X,Z)} \right\rvert f_{X,Y}$$

$$= \left\lvert \det \begin{bmatrix} \underbrace{\frac{\partial X}{\partial X}}_{=1} & \frac{\partial X}{\partial Z} \\ \underbrace{\frac{\partial Y}{\partial X}}_{=0} & \frac{\partial Y}{\partial Z} \end{bmatrix} \right\rvert f_{X,Y}$$

$$= f_{X,Y} \left\lvert \frac{\partial y}{\partial z} \right\rvert$$

Convolution formula

let $Z(X,Y)$ be a differentiable function of X, Y

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Z}(x,z)\, dx = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left\lvert \frac{\partial y}{\partial z} \right\rvert dx$$

ex (Convution formula for sum)

Let $Z(x,y) = x+y$

Find the convolution formula for Z.

Step 1 Solve for Y treating X as a fixed constant

$$Y = Z - X$$

Step 2 Find $\frac{dY}{dZ}$

$$\frac{dy}{dZ} = 1 - \underbrace{\frac{dx}{dZ}}_{=0} = 1$$

Step 3 substitute y, $\frac{\partial y}{\partial z}$ in $f_Z(z) = \int_{x=-\infty}^{\infty} f_{x,y}(x,y) \left|\frac{\partial y}{\partial z}\right| dx$

$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{x,y}(x, z-x)\, dx$$

← convolution formula for sum.

If X and Y are indep, $f_{x,y}(x, z-x) = f_x(x) f_y(z-x)$

tinyurl.com/novem-2022

Stat 134

Friday October 21 2022

1. Let X and Y be iid $Exp(\lambda)$ (recall $f_X(x) = \lambda e^{-\lambda x}$). Find the density of $Z = X + Y$ using the convolution formula for sum

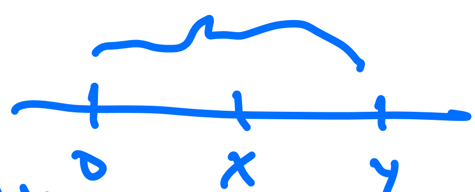
$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{(X,Y)}(x, z-x)dx$$

a $f_Z(z) = \lambda^2 e^{-\lambda(z-2x)}$

b $f_Z(z) = \lambda^2 z e^{-\lambda z}$

c $f_Z(z) = \lambda^2 z^2 e^{-2\lambda z}$

d none of the above



$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, z-x)\,dx = \int_0^z f_X(x)\, f_Y(z-x)\,dx$$

z fixed

$$= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\,dx$$

$$= \lambda^2 e^{-\lambda z}\int_0^z dx = \lambda^2 e^{-\lambda z} \cdot x\Big|_0^z = \boxed{\lambda^2 z e^{-\lambda z}}$$

$Z \sim Gamma(2, \lambda)$ ✓

② Sec 5.4 Triangular density

Let $\left.\begin{array}{l} X \sim \text{Unif}\{0,1,2,\dots,6\} \\ Y \sim \text{Unif}\{0,1,2,\dots,6\} \end{array}\right\}$ indep.

Find probability mass function of $Z = X + Y$

<u>$0 \le Z \le 6$</u>

$$P(Z=4) = \underbrace{P(0,4)}_{=\left(\frac{1}{7}\right)^2} + \underbrace{P(1,3)}_{=\left(\frac{1}{7}\right)^2} + P(2,2) + P(3,1) + P(4,0) = 5/49$$

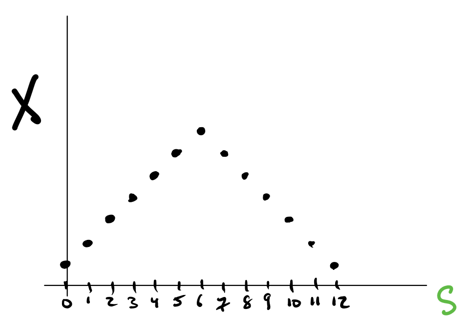
$$P(Z=z) = \sum_{x=0}^{z} P(X=x, Y=z-x)$$

<u>$7 \le Z \le 12$</u>

$$P(Z=8) = P(2,6) + P(3,5) + \dots + P(6,2) = 5/49$$

$$P(Z=z) = \sum_{x=z-6}^{6} P(x, z-x)$$

The distribution of $Z = X + Y$ looks like



Continuous case:

$X \sim U(0,1)$
$Y \sim U(0,1)$ } indep

Find density of $Z = X + Y$

$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, z-x)\, dx = \int_{x=-\infty}^{x=\infty} 1_{x\in(0,1)}\, 1_{z-x\in(0,1)}\, dx$$

range of values of Z? 0 - 2

for $0<z<1$, x can't be bigger than z
$0<x<z$

$$f_Z(z) = \int_0^z 1\, dx = z.$$

if $z = 1.25$, x can't be smaller than $.25$
$.25 < x < 1$

if $z = 1.5$, x can't be smaller than $.5$
$.5 < x < 1$

if $1<z<2$, $z-1 < x < 1$

$$f_Z(z) = \int_{x=z-1}^{1} 1\,dx = 1-(z-1) = 2-z$$

So $f_Z(z) = \begin{cases} z & 0<z<1 \\ 2-z & 1<z<2 \end{cases}$