Stat 134 lec 32

Warmup 9:00–9:10

Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.

The joint density for (X,Y) is:

$$f_{X,Y}(x,y) = \binom{10}{6,1,1,1,1} x^6 (y-x)(1-y) \text{ where for } 0<x<y<1.$$

let $Z = Y - X$

1) Solve for Y treating X as a constant $\quad Y = Z + X$

2) Find $\frac{dY}{dZ} = 1$

3) Find the convolution formula for $Z = Y - X$

$$\int_{-\infty}^{\infty} f_{X,Y}(x, z+x)\, dx$$

recall:

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left|\frac{\partial y}{\partial z}\right| dx$$

fixed

For a fixed z, what is the largest value of x?

$1 = x + z$

$\Rightarrow x = 1 - z$

$$= \boxed{\int_{x=0}^{x=1-z} f_{X,Y}(x, z+x)\, dx}$$

Announcement: MT2 Friday 11/18 (take home)
MGF, chap 4 (skip sec 4.3),
Chap 5,
review materials coming.

Last time

Sec 5.4

Convolution formula

let $Z(X,Y)$ be a differentiable function of X, Y

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left| \frac{\partial y}{\partial z} \right| dx$$

In particular if $Z = X + Y$

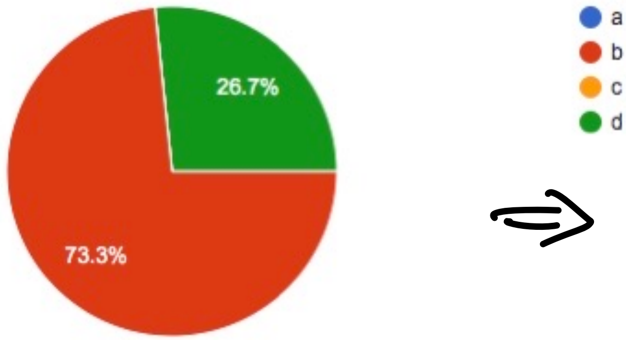
$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx$$

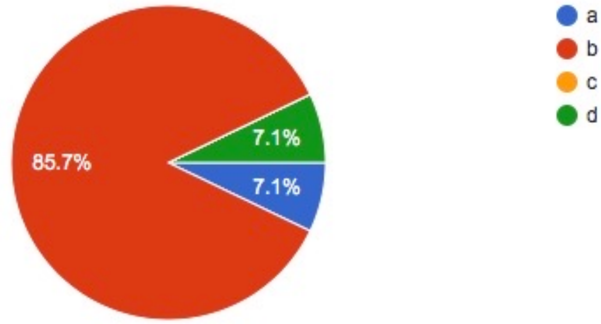
ex (triangular density)

let $X, Y \overset{iid}{\sim} U(0,1)$

$Z = X+Y$

$$f_Z(z) = \begin{cases} \int_0^z f_{X,Y}(x, z-x)\, dx = z & \text{for } 0<z<1 \\ \int_{z-1}^1 f_{X,Y}(x, z-x)\, dx = 2-z & \text{for } 1<z<2 \end{cases}$$





Friday October 21 2022

1. Let X and Y be iid $Exp(\lambda)$ (recall $f_X(x) = \lambda e^{-\lambda x}$). Find the density of $Z = X + Y$ using the convolution formula for sum

$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{(X,Y)}(x, z-x)dx$$

a $f_Z(z) = \lambda^2 e^{-\lambda(z-2x)}$

b $f_Z(z) = \lambda^2 z e^{-\lambda z}$

c $f_Z(z) = \lambda^2 z^2 e^{-2\lambda z}$

d none of the above

b	As X and Y are independent, we can apply the convolution formula and then split the joint density.
b	as z is fixed, for z-x to be positive, upper bound of x should be z. thus by formula we have integral 0 to z： λ² e^(-λz) dx. then answer is λ² e^(-λz)z which is B

ex (Convolution of difference)

$X = U_{(7)}$, $Y = U_{(9)}$ of 10 iid $U(0,1)$

$Z = Y - X$

The joint density $f_{X,Y}(x,y) = c\, x^6 (y-x)(1-y)$ where $c = \binom{10}{6,1,1,1,1}$ for $0<x<y<1$.

$$f_Z(z) = \int_0^{1-z} f_{X,Y}(x, z+x)\,dx$$

Find the density of $Z = Y - X$
what distribution is Z?

$$c = \binom{10}{6,1,1,1,1}$$

$$f_Z(z) = \int_0^{1-z} c\, x^6 (x+z-x)(1-(x+z))\,dx$$

$$= cz \int_0^{1-z} \left((1-z)x^6 - x^7\right) dx$$

$$= cz \left[(1-z)\frac{x^7}{7} - \frac{x^8}{8} \right]\Bigg|_{x=0}^{x=1-z}$$

$$= cz\left(\frac{(1-z)^8}{7} - \frac{(1-z)^8}{8}\right) = \boxed{\frac{c}{56} z(1-z)^8}$$

$$Z \sim \text{Beta}(2, 9)$$

$$U_{(9)} - U_{(7)} = U_{(2)} \sim \text{Beta}(2, \underbrace{10-2+1}_{=9})$$

recall $U_{(k)} \sim \text{Beta}(k, n-k+1)$

__Today__ (1) (see #13 p 355) Uniform Spacing

(2) Sec 5.4 More Convolution Formulas

① (see #13 p 355) <u>Uniform Spacing</u>

We saw above

Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.

then $Z = Y - X \sim \text{Beta}(2,9)$

We know $U_{(9)} - U_{(7)}$ and $U_{(2)}$

both are $\text{Beta}(2,9)$

<u>More generally</u> (Uniform Spacing)

You randomly throw n dart at $[0,1]$.

For $0 < a < a+k \leq n$, $U_{(a+k)} - U_{(a)}$ is ?

$U_{(k)} \sim \text{Beta}(k, n-k+1)$

(3) Other convolution formulas

ex Let $Z = \frac{Y}{X}$. Find the convolution formula for Z.

Step 1 $Y = XZ$

Step 2 $\frac{dy}{dz} = X$

Step 3 $f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, xz)\,|x|\,dx$

$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left|\frac{\partial y}{\partial z}\right| dx$

Stat 134

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1. Let X and Y be iid $Exp(1)$ (recall $f_X(x) = e^{-x}$). Find the density of $Z = \frac{Y}{X}$ using the convolution formula

$$f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{(X,Y)}(x, zx)|x|dx$$

a $f_Z(z) = \frac{1}{(1+z)}$ for $0 < z < \infty$

b $f_Z(z) = \frac{1}{(1+z)^2}$ for $0 < z < \infty$

c $f_Z(z) = \frac{1}{2(1+z)^2}$ for $0 < z < \infty$

d none of the above

$$f_Z(z) = \int_{x=0}^{\infty} \underbrace{f_X(x)}_{e^{-x}} \underbrace{f_Y(zx)}_{e^{-zx}} x\,dx$$

$$= \int_0^\infty x e^{-(1+z)x} dx$$

variable part of Gamma $(r=2, \lambda=1+z)$

$X \sim Gamma(r, \lambda)$

$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$

$$= \frac{1}{\text{constant part of Gamma}(2, 1+z)} = \frac{\overset{=1}{\Gamma(2)}}{(1+z)^2} \text{ for } 0<z<\infty$$

Ex Let $Z = \frac{X}{X+Y}$. Find the convolution formula for Z.

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left| \frac{\partial y}{\partial z} \right| dx$$

$$zx + zy = x \Rightarrow zy = x - zx$$

$$\Rightarrow y = \frac{x(1-z)}{z}$$

$$\frac{dy}{dz} = x \left[\frac{(1-z)'z - (1-z)z'}{z^2} \right]$$

$$= x \left[\frac{-z - 1 + z}{z^2} \right] = -\frac{x}{z^2}$$

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(x, \frac{x(1-z)}{z}\right) \frac{|x|}{z^2}\, dx}$$

extra

Let $X \sim U(0,1)$ and $Y \sim U(0,1)$ be independent. The density of $Z = Y/X$

Z takes values in $(0,\infty)$

$0<z<1$

$$f(z) = \int_0^1 1_{x\in(0,1)} \cdot 1_{xz\in(0,1)} \cdot x\,dx$$

$$= \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{1/2} \quad \text{for } 0<z<1$$

$z>1$

$$f(z) = \int_0^{\frac{1}{z}} 1_{x\in(0,1)}\, 1_{xz\in(0,1)}\, x\,dx$$

$$= \int_0^{1/z} x\,dx = \frac{x^2}{2}\Big|_0^{1/z} = \boxed{\frac{1}{2z^2}}$$

for $z>1$