Stat 134 lec 33

Warmup 11:00 – 11:10

ex Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.
The joint density $f_{X,Y}(x,y) = c\,x^6 (y-x)(1-y)$ where $c = \binom{10}{6,1,1,1,1}$ for $0<x<y<1$.

Find the density of $Z = Y - X$
what distribution is Z?

Hint: Use the convolution formula $f_Z(z) = \int_0^{1-z} f(x, x+z)\,dx$

$$c = \binom{10}{6,1,1,1,1}$$

$$f_Z(z) = \int_0^{1-z} c\,x^6 (x+z-x)(1-(x+z))\,dx$$

$$= cz \int_0^{1-z} \left((1-z)x^6 - x^7\right) dx$$

$$= cz \left[(1-z)\frac{x^7}{7} - \frac{x^8}{8} \right]\Bigg|_{x=0}^{x=1-z}$$

$$= cz \left(\frac{(1-z)^8}{7} - \frac{(1-z)^8}{8} \right) = \frac{c}{56} z (1-z)^8.$$

$$Z \sim \text{Beta}(2, 9)$$

$U_{(9)} - U_{(7)}$ should have the same distribution as

$U_{(k)} \sim \text{Beta}(k, n-k+1)$

$$U_{(2)} - 0 = U_{(2)} \sim \text{Beta}(2, \underbrace{10-2+1}_{9}) \checkmark$$

Announcement: MT2 Friday 11/19 (take home)
approx 50 min. M6F, chap 4 (skip sec 4.3),
Chap 5,
review materials coming.

Last time

Sec 5.4 Density Convolution Formula of $S=X+Y$

Assume $X>0, Y>0$

$$f_S(s) = \int_{x=0}^{x=s} f(x, s-x)\, dx$$

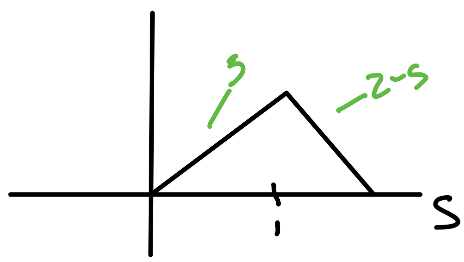
convolution formula for densities.

ex (triangular density)

let $X, Y \overset{iid}{\sim} U(0,1)$

$S = X+Y$

$$f_S(s) = \begin{cases} s & \text{for } 0<s<1 \\ 2-s & \text{fo } 1<s<2 \end{cases}$$



Convolution formula for $Z=Y-X$ for $0<X<Y$

$$f_Z(z) = \int_0^{1-z} f(x, x+z)\, dz$$

Today
1. (see #13 p 355) Uniform Spacing
2. Sec 5.4 More Convolution Formulas
3. Sec 6.1, 6.2 Conditional Distribution, Expectation — discrete case
4. Sec 5.4 General convolution formula

① (see #13 p 355) <u>Uniform Spacing</u>

We saw above

Let $X \sim U_{(7)}$, $Y \sim U_{(9)}$ for 10 iid $U(0,1)$.

then $Z = Y - X \sim \text{Beta}(2,9)$

We know $U_{(9)} - U_{(7)}$ and $U_{(2)}$

both are Beta $(2,9)$

<u>More generally</u> (Uniform Spacing)

You randomly throw n dart at $[0,1]$.

For $0 < a < a+k \leq n$, $U_{(a+k)} - U_{(a)}$ is ?

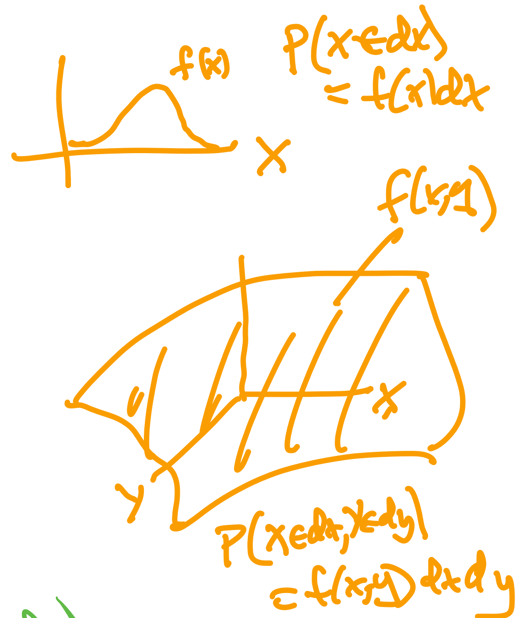
$U_{(k)} \sim \text{Beta}(k, n-k+1)$

② Convolution formula for density of ratio Y/X

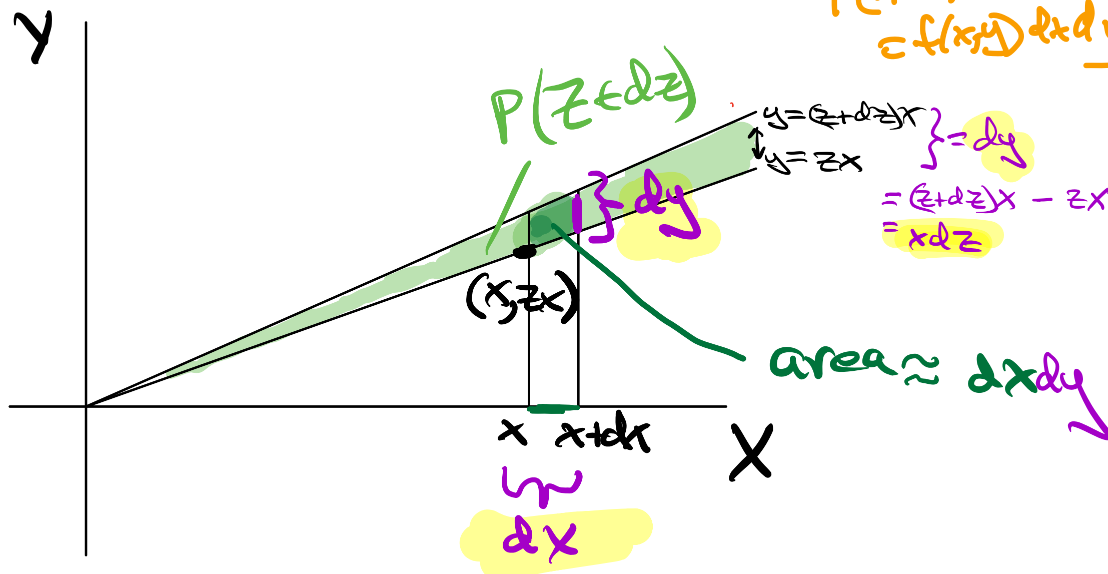
$X>0,\ Y>0$

let $Z = \dfrac{Y}{X}$.

Find $f_Z(z)$.



Picture $\quad Y = ZX$ — slope



$$P(Z\in dz) = \int_{x=0}^{x=\infty} P(Z\in dz, X\in dx)$$

$$\| \qquad\qquad dy = x\,dz$$

$$f_Z(z)dz \qquad = \int_{x=0}^{x=\infty} f(x, zx)\, dy\, dx$$

Convolution formula.

$$\Rightarrow f_Z(z) = \int_{x=0}^{x=\infty} f(x, zx)\, x \, dx = \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\, x \, dx$$ - if X, Y indep.

Ex

Let $X, Y \overset{iid}{\sim} \text{Exp}(1)$. $Z = \frac{Y}{X}$ $\quad f_X(x) = e^{-x}$

Find $f_Z(z)$.

Hint: use Convolution formula

$$f_Z(z) = \int_{x=0}^{\infty} \underbrace{f_X(x)}_{= e^{-x}} \underbrace{f_Y(zx)}_{= e^{-zx}}\, x \, dx$$

$$= \int_0^{\infty} x e^{-(1+z)x} \, dx$$

variable part of Gamma $(r=2, \lambda = 1+z)$

$X \sim \text{Gamma}(r, \lambda)$

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$$

$$= \frac{1}{\underbrace{\text{constant part of Gamma}(2, 1+z)}_{\frac{(1+z)^2}{\Gamma(2)}}} = \frac{\overset{=1}{\Gamma(2)}}{(1+z)^2} \text{ for } 0 < z < \infty$$

③ Sec 6.1 Conditional Distribution: Discrete case.

let X, N discrete RVs w/ joint distribution $P(X=x, N=n)$.

Bayes rule

$$P(X=x \mid N=n) = \frac{P(X=x, N=n)}{P(N=n)}$$

$$\Rightarrow \quad P(X=x, N=n) = P(X=x \mid N=n) P(N=n)$$

Rule of average conditional probabilities

marginal prob of X

$$P(X=x) = \sum_n P(X=x, N=n)$$

$$= \sum_n P(X=x \mid N=n) P(N=n)$$

ex

\# busses in 1 min

\# green busses in 1 min

Let N have Poisson (λ) distribution. Let X be a random variable with the following property: for every n, the conditional distribution of X given $(N=n)$ is binomial (n,p). Find the unconditional distribution of X and state its parameter(s). Show all your work for full credit.

X = # green busses in 1 min

p = prob a bus is green

N = superposition of 2 poisson processes

$\Rightarrow X \sim \text{Pois}(\lambda p)$ by poisson thinning.

$$P(N=n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$P(X=x \mid N=n) = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Find $P(X=x)$

$$P(X=x) = \sum_{n=x}^{\infty} P(X=x \mid N=n) P(N=n)$$

$$= \sum_{n=x}^{\infty} \frac{n!}{x!(n-x)!} p^x q^{n-x} \frac{e^{-\lambda}\lambda^n}{n!} \qquad \lambda^x \cdot \lambda^{n-x}$$

$$= \frac{e^{-\lambda}\lambda^x p^x}{x!} \boxed{\sum_{n=x}^{\infty} \frac{\lambda^{n-x} q^{n-x}}{(n-x)!}} = \underbrace{1 + \lambda q + \frac{(\lambda q)^2}{2!} + \dots}_{= e^{\lambda q}}$$

Finish

$$\frac{e^{-\lambda(1-q)} (\lambda p)^x}{x!} \quad p$$

$X \sim \text{Pois}(\lambda p)$ ✓