Stat 134 lec 34

Warmup 11:00-11:10

8 transistors (type 1) are distributed $Exp\left(\frac{1}{100}\right)$ and 4 transistors (type 2) are $Exp\left(\frac{1}{200}\right)$.
Let T be the lifetime of a randomly picked transistor.

a) Find $E(T \mid \text{transistor is type } 1)$

b) Find $E(T)$

a) $T \mid \text{type } 1 \sim Exp\left(\frac{1}{100}\right)$

$E(T \mid \text{type } 1) = 100$

b) let X = type of transistor

$E(T) = E(E(T \mid X))$

$E(T) = E(T \mid X=1) \cdot P(X=1) + E(T \mid X=2) P(X=2)$

$= 100 \cdot \frac{8}{12} + 200 \cdot \frac{4}{12} = \boxed{133.3}$

Last time

Sec 5.4 Uniform Spacing

You ran only throw n dart at $[0,1]$.

For $0 < a < a+k \le n$, $U_{(a+k)} - U_{(a)} = U_{(k)} = \text{Beta}(k, n-k+1)$

Sec 5.4 Convolution formula for density of ratio Y/X

$X > 0$, $Y > 0$

let $Z = \frac{Y}{X}$.

Convolution formula.

$$f_Z(z) = \int_{x=0}^{x=\infty} f(x, zx)\, x\, dx = \int_{x=0}^{x=\infty} f_X(x) f_Y(zx)\, x\, dx$$

- if X, Y indep.

Sec 6.1

Rule of average conditional probabilities (discrete case)

Let X and N be discrete RV w/ joint distribution $P(X=x, N=n)$.

$$P(X=x) = \sum_n P(X=x, N=n)$$

$$= \sum_n P(X=x \mid N=n) P(N=n)$$

Today

1. Sec 5.4 General convolution formula
2. Sec 6.2 Properties of conditional expectation.
3. Sec 5.4 Uniform Spacing continued

(1) sec 5.4 General convolution formula

We have different convolution formulas for sums and quotients.

We can write a general convolution formula for any operation.

1 dimensional change of variables

RV	transformed RV
Y	$Z(Y)$ — a differentiable function ex $Z = Y^3$
f_Y	$f_Z = \left\lvert \frac{\partial y}{\partial z} \right\rvert f_Y$ $\frac{1}{\left\lvert \frac{\partial z}{\partial y} \right\rvert}$

2 dimensional change of variables

RV	transformed RV
(X, Y)	$(X, Z(X,Y))$ — a differentiable function ex $Z = X+Y$ ex $Z = \frac{X}{Y}$
$f_{X,Y}$	$f_{X,Z} = \left\lvert \det \frac{\partial(x,y)}{\partial(x,z)} \right\rvert f_{X,Y}$

$$= \left\lvert \det \begin{bmatrix} \overset{=1}{\frac{\partial x}{\partial x}} & \frac{\partial x}{\partial z} \\ \underset{=0}{\frac{\partial y}{\partial x}} & \frac{\partial y}{\partial z} \end{bmatrix} \right\rvert f_{X,Y}$$

$$= f_{X,Y} \left\lvert \frac{\partial y}{\partial z} \right\rvert$$

Convolution formula

let $Z(X,Y)$ be a differentiable function of X, Y

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Z}(x,z)\, dx = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y) \left\lvert \frac{\partial y}{\partial z} \right\rvert dx$$

ex Let $Z = \frac{Y}{X}$. Find the convolution formula for Z,

$\Rightarrow Y = XZ \Rightarrow \frac{\partial y}{\partial z} = x$

$\Rightarrow f_Z(z) = \int_{x=-\infty}^{x=\infty} f_{X,Y}(x, xz)\,|x|\,dx$

Convolution formula for quotient.

ex Let $Z = \frac{X}{X+Y}$. Find the convolution formula for Z,

$$f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\left|\frac{\partial y}{\partial z}\right| dx$$

Step 1 solve for Y

$zx + zy = x \Rightarrow zy = x - zx$

$\Rightarrow \boxed{y = \frac{x(1-z)}{z}}$

Step 2 Find $\frac{\partial y}{\partial z}$

$$\frac{\partial y}{\partial z} = x\left[\frac{(1-z)'z - (1-z)z'}{z^2}\right] = x\left[\frac{-z-1+z}{z^2}\right] = \boxed{\frac{-x}{z^2}}$$

Step 3 substitute y, $\frac{\partial y}{\partial z}$ in $f_Z(z) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\left|\frac{\partial y}{\partial z}\right| dx$

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}\left(x, \frac{x(1-z)}{z}\right)\frac{|x|}{z^2}\,dx}$$

Sec 6.2 Conditional Expectation
(discrete case)

Bayes rule:

recall $P(T=t \mid S=s) = \dfrac{P(T=t, S=s)}{P(S=s)}$

ex (T,S) is joint distribution below.

Find $P(T=3 \mid S=7)$

$$= \frac{P(T=3, S=7)}{P(S=7)} = \frac{.3}{.4} = \boxed{.75}$$

	T=3	T=4	Sum ← marginal of S
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

marginal of T

Find $P(T=4 \mid S=7)$

$$\frac{P(T=4, S=7)}{P(S=7)} = \frac{.1}{.4} = \boxed{.25}$$

Find $E(T \mid S=7)$

$$\sum_{t \in T} t\,P(T=t \mid S=7) = 3 \cdot P(T=3 \mid S=7) + 4\,P(T=4 \mid S=7)$$

$$= 3(.75) + 4(.25) = \boxed{3.25}$$

Find $E(T \mid S=6)$

$$3 \cdot P(T=3 \mid S=6) + 4 \cdot P(T=4 \mid S=6)$$

$$= 3\left(\frac{.2}{.4}\right) + 4\left(\frac{.2}{.4}\right) = \boxed{3.5}$$

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1.0

← marginal of S

marginal of T

$$\left.\begin{aligned} E(T \mid S=7) &= 3.25 \\ E(T \mid S=6) &= 3.5 \\ E(T \mid S=5) &= 3.5 \end{aligned}\right\} \text{function of } S$$

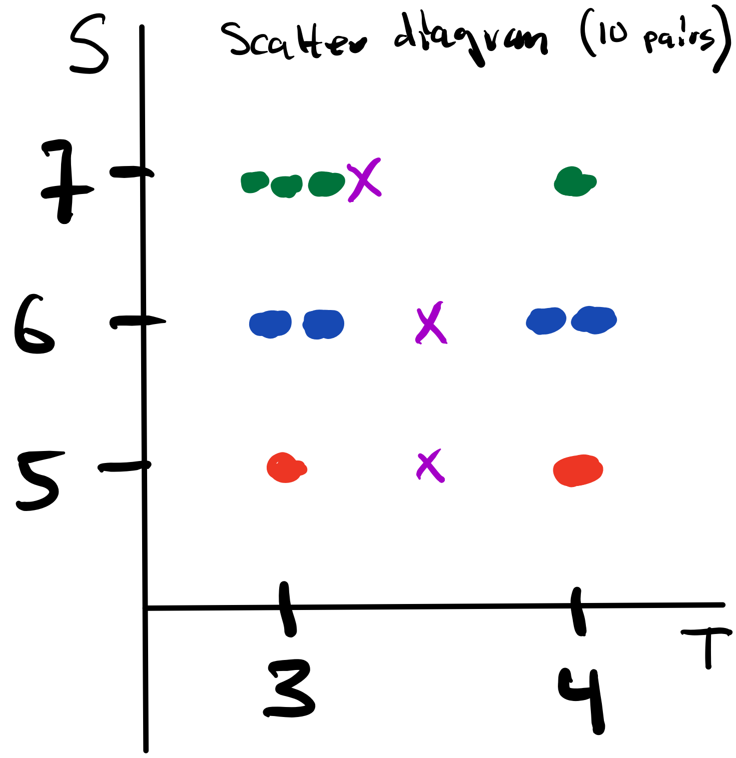
Picture

joint distribution

	T=3	T=4	sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
sum	0.6	0.4	1

$\longleftrightarrow$

Scatter diagram (10 pairs)



Two main points:

1. E(T|S) is a function of S.
2. E(T|S) is a RV so it has an expectation.

Next we explore the expectation of E(T|S).

Let $g(s) = E(T|S)$

$g(7) = E(T|S=7) = 3.25$

$g(6) = 3.5$

$g(5) = 3.5$

$$E(g(S)) = \sum_{s \in S} g(s) P(S=s)$$

$$= 3.25(.4) + 3.5(.4) + 3.5(.2)$$

$$= 3.4$$

	T=3	T=4	Sum
S=7	0.3	0.1	0.4
S=6	0.2	0.2	0.4
S=5	0.1	0.1	0.2
Sum	0.6	0.4	1

Find $E(T) = \sum_{t \in T} t P(T=t) = 3(.6) + 4(.4)$

$= 3.4$

In other words,

$$E(E(T|S)) = E(T)$$

This is called the property of iterated expectations.

Intuitively,

If you have a class that is $2/3$ girls and $1/3$ boys and the girls weigh on average 100 lbs and boys weigh 200 lbs then the average weight of the class should be $\frac{2}{3}(100) + \frac{1}{3}(200)$. i.e. we take the weighted average of the averages.

Rule of average conditional expectations

For any random variable T with finite expectation and any discrete RV S,

$$E(T) = \sum_{\text{all } s} E(T \mid S=s) \cdot P(S=s)$$

(see end of this lecture for a formal proof)

③ Uniform spacing continued

let $U_1, \dots, U_{10} \overset{iid}{\sim} \text{Unif}(0,1)$

and $U_{(1)}, \dots, U_{(10)}$ be ordered standard uniform.

let $Y \sim U_{(7)}$, $X \sim U_{(9)}$

Let $Z = \frac{Y}{X}$

What distribution is Z?

$$0 = \frac{0}{U_{(9)}}, \frac{U_{(1)}}{U_{(9)}}, \frac{U_{(2)}}{U_{(9)}}, \dots, \frac{U_{(9)}}{U_{(9)}} = 1$$

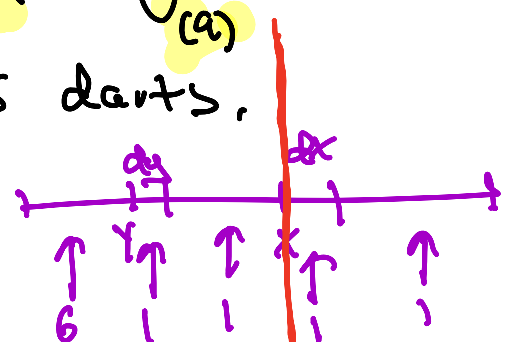
Since $U_{(9)}$ is constant $\frac{U_{(1)}}{U_{(9)}}, \dots, \frac{U_{(8)}}{U_{(9)}}$ is an ordering of 8 points between 0 and 1.

So $Z = \frac{Y}{X} = \frac{U_{(7)}}{U_{(9)}}$ is the 7^{th} order statistic out of 8 darts.

so $Z = U_{(7)}$ out of 8

$\sim \text{Beta}(7, \underbrace{8-7+1}_{=2})$

Picture



7^{th} ordered uniform out of 8 darts before new 1 ex

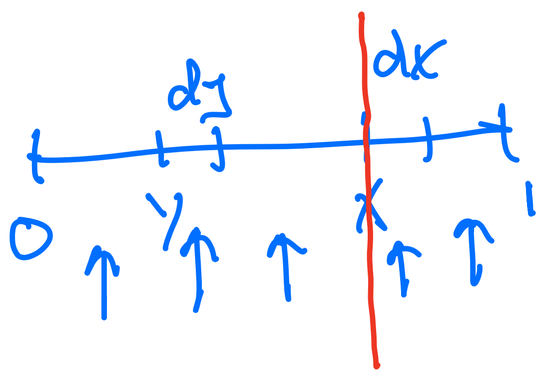
Throw down 5 darts on $(0,1)$.

$Y = U_{(2)} \quad X = U_{(4)}$

a) What distribution is $\frac{Y}{X}$?

b) Find $P(X > 4Y)$

$$\frac{Y}{X} = \frac{U_{(2)} \text{ out of } 5}{U_{(4)} \text{ out of } 5} = U_{(2)} \text{ out of } 3 \sim \text{Beta}(2, \underbrace{3-2+1}_{=2})$$



$$\frac{U_{(1)}}{U_{(4)}} \quad \frac{U_{(2)}}{U_{(4)}} \quad \frac{U_{(3)}}{U_{(4)}}$$
$$\parallel \quad \parallel \quad \parallel$$
$$U_{(1)} \quad U_{(2)} \quad U_{(3)}$$

$$P(X > 4Y) = P\left(\frac{Y}{X} < \frac{1}{4}\right)$$

$$z = \frac{Y}{X} \sim \text{Beta}(2,2)$$

$$f_Z(z) = \frac{\Gamma(4)}{\Gamma(2)\Gamma(2)} z(1-z) = 6z(1-z)$$

$$P(Z < 1/4) = 6\int_0^{1/4} z\,dz - 6\int_0^{1/4} z^2\,dz = \boxed{10/64}$$

Appendix

Iterated Expectation

We show $E(Y) = E(E(Y|X))$:

$$E(Y) = \sum_{\text{all } y} y P(Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(X=x, Y=y)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} \frac{P(X=x, Y=y)}{P(X=x)} P(X=x)$$

$$= \sum_{\text{all } y} y \sum_{\text{all } x} P(Y=y \mid X=x) P(X=x)$$

$$= \sum \quad \sum y P(Y=y \mid X=x) \cdot P(X=x)$$

$$= \sum_{\text{all } x} E(Y \mid X=x) \cdot P(X=x)$$

$$= E(E(Y|X))$$

$\square$