Stat 134 Lec 35

Warmup: 11:00–11:10

Let $N \sim \text{Geom}(p)$ on $1, 2, 3, \dots$

Suppose $X \mid N = n \sim \text{Unif}(0, 1, 2, \dots, n)$

Find $E(X)$ Hint $E(X \mid N) = \frac{N}{2}$

$$E(X) = \sum_{n=1}^{\infty} \underbrace{E(X \mid N=n)}_{=\frac{n}{2}} \underbrace{P(N=n)}_{=q^{n-1}p} = \frac{p}{2} \sum_{n=1}^{\infty} n q^{n-1} = \frac{1}{p^2} = \boxed{\frac{1}{2p}}$$

$$\underbrace{E(N)}_{=\frac{1}{p}} = \sum_{n=1}^{\infty} n p q^{n-1} = p \sum_{n=1}^{\infty} n q^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} n q^{n-1} = \frac{1}{p^2}$$

Another way

$$E(X) = E(\underbrace{E(X \mid N)}_{=\frac{N}{2}}) = E\left(\frac{N}{2}\right) = \frac{1}{2} \underbrace{E(N)}_{=\frac{1}{p}} = \boxed{\frac{1}{2p}}$$

Announcement:

Schedule this week!

M: regular lecture

W: MT2 review

F: regular lecture (MT2 available Friday 6pm – due Sunday 6pm)

Last time

Sec 6.2 Rule of iterated expectation

For any random variable T with finite expectation and any discrete RV S,

$$E(T) = E(E(T|S)) = \sum_{\text{all } s} E(T|S=s) \cdot P(S=s)$$

Today

1. Sec 6.2 Properties of conditional expectation
2. Sec 6.3 Conditional density
3. Sec 6.3 Bayesian statistics

① Sec 6.2 Properties of conditional expectation

$(Y+Z)|X=x \;=\; Y|X=x \;+\; Z|X=x$ so

$$E(Y+Z|X=x) = E(Y|X=x) + E(Z|X=x)$$

What is $E(X+Z|X=5) = ?$

$$E(X|X=5) + E(Z|X=5)$$

$$\underset{5}{\parallel}$$

Properties

1. $E(X) = E(E(X|Y))$ equality of numbers
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$

equality of RVs

Notation

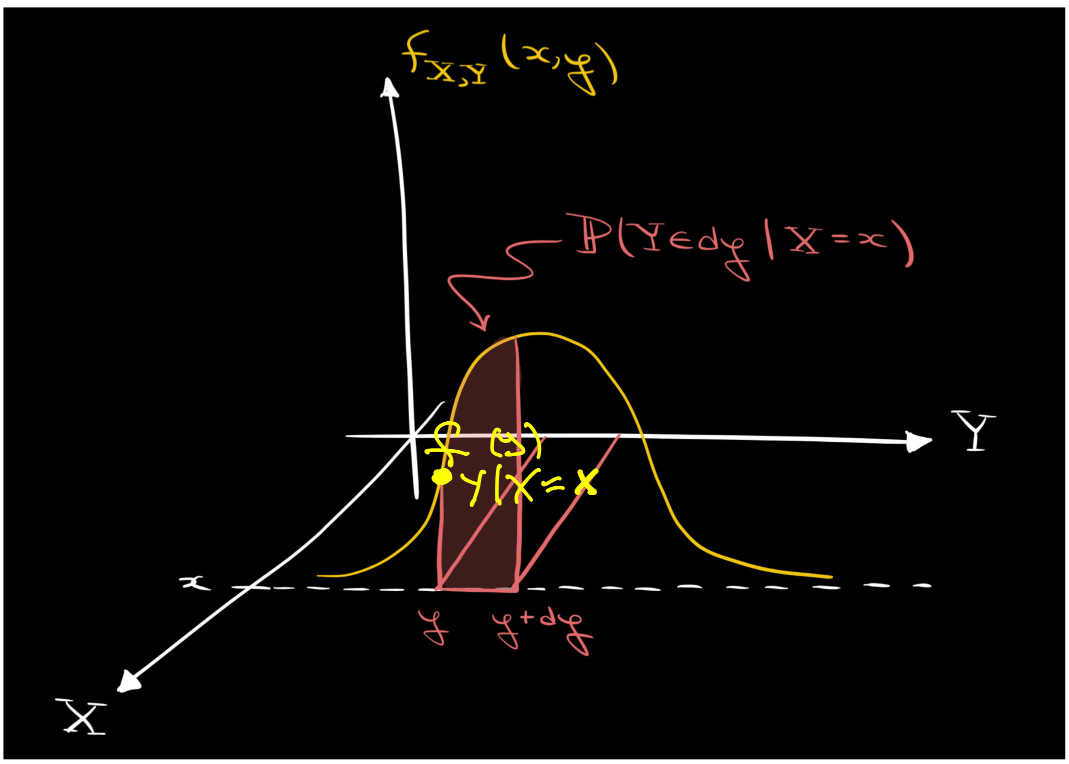
If (x,y) is joint discrete
$P(Y|X=x)$ is conditional prob,
If (x,y) is joint continuous
$f_{Y|X=x}(y)$ is conditional density.

② sec 6.3 Conditional Density:

Let X, Y be continuous RVs with joint density $f_{X,Y}(x,y)$

Let $f_{Y|X=x}(y)$ be a slice of $f_{X,Y}(x,y)$ through $X=x$.

Define $P(Y \in dy \mid X=x)$ as the area under $f_{Y|X=x}(y)$ for $Y \in dy$



By Baye's rule,

$$P(Y\in dy \mid X=x) = \lim_{dx\to 0} \frac{P(Y\in dy,\, X\in dx)}{P(X\in dx)}$$

$\approx f(x,y)\,dx\,dy$

$\approx f_X(x)\,dx$

$= f_{Y|X=x}(y)\,dy$

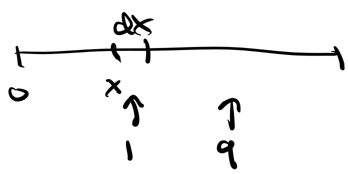
$$\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$$

Conditional density of Y given $X=x$

ex Let $U_1,\dots,U_{10} \overset{iid}{\sim} U(0,1)$

$X=U_{(1)}$, $Y=U_{(10)}$

$f(x,y) = 90\,(y-x)^8$



$$f_X(x) = \binom{10}{1,9}(1-x)^9 = 10(1-x)^9$$

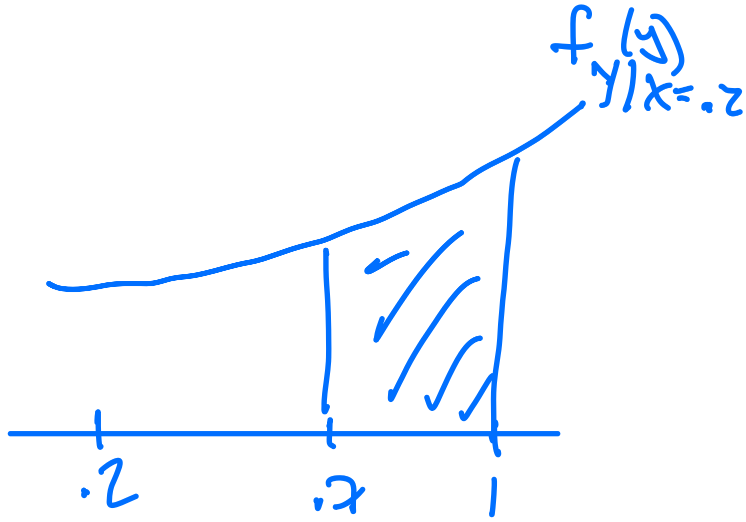
Find $P(Y>.7 \mid X=.2)$

$$f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}$$

a) Find $f_{Y|X=.2}(y)$

$$f_{Y|X=.2}(y) = \frac{f(.2,y)}{f_X(.2)} = \boxed{\frac{90(y-.2)^8}{10(.8)^9}}$$

b) Find $\int_{.7}^{1} f_{Y|X=.2}(y)\,dy$



$$\frac{9}{(.8)^9}\int_{.7}^{1}(y-.2)^8\,dy = \frac{9}{(.8)^9}\int_{u=.5}^{u=.8} u^8\,du$$

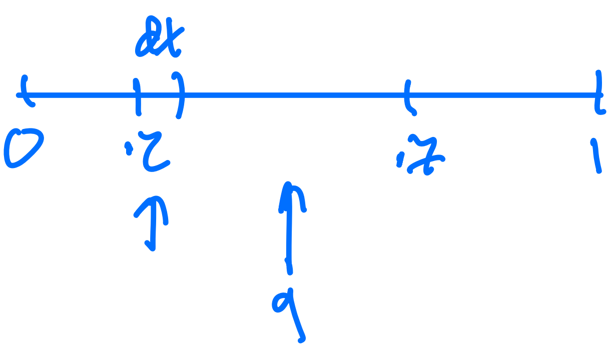
let $u = y-.2$

$$= \frac{9}{(.8)^9}\,\frac{u^9}{9}\Big|_{.5}^{.8} = \boxed{1-\left(\frac{.5}{.8}\right)^9}$$

Alternatively
use fact $X = U_{(1)}$, $Y = U_{(10)}$

$$P(Y \ge .7 \mid X = .2) = \lim_{dx \to 0} \frac{P(Y \ge .7,\ X \in .2 + dx)}{P(X \in .2 + dx)}$$

$$= 1 - \lim_{dx \to 0} \frac{P(Y \le .7,\ X \in .2 + dx)}{P(X \in .2 + dx)}$$



$$P(Y \le .7,\ X \in .2 + dx) = \binom{10}{1,9} 1\,dx\,(.7 - .2)^9$$

$$P(X \in .2 + dx) = \binom{10}{1,9} 1\,dx\,(1 - .2)^9$$

$$1 - \lim_{dx \to 0} \frac{10\,\cancel{dx}\,(.5)^9}{10\,\cancel{dx}\,(.8)^9} = \boxed{1 - \left(\frac{.5}{.8}\right)^9}$$

Rule of average conditional probabilities (discrete case)

Let X and Y be discrete RVs w/ joint distribution $P(X=x, Y=y)$

$$P(Y=y) = \sum_x P(Y=y, X=x) = \sum_x P(Y=y \mid X=x) P(X=x)$$

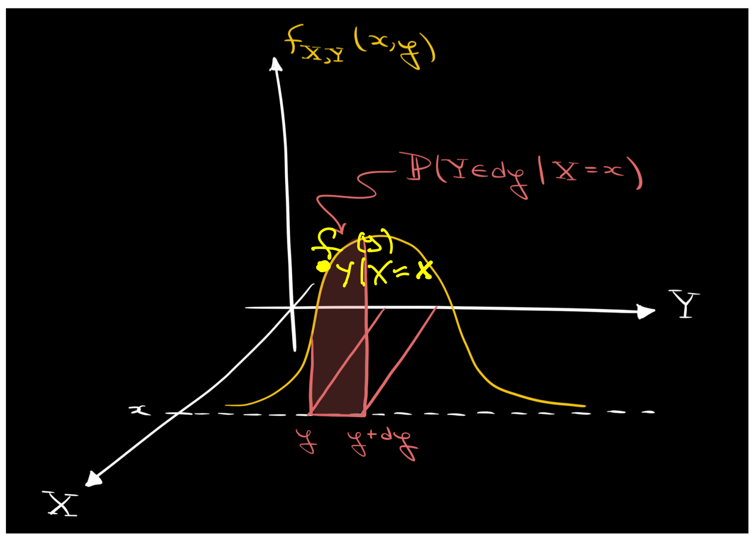
Rule of average conditional probabilities (Continuous case)

Let X and Y be continuous RVs w/ joint distribution $f(x,y)$

$$P(Y \in dy) = \int_{x \in X} P(Y \in dy, X=x)\,dx$$

$$= \int_{x \in X} P(Y \in dy \mid X=x) f_X(x)\,dx$$

$$= \int_{x \in X} f_{Y|X=x}(y)\,dy\, f_X(x)\,dx$$



ex $X \sim \mathrm{Unif}(0,1)$

$I_1 \mid X=x,\ I_2 \mid X=x \overset{iid}{\sim} \mathrm{Ber}(x)$

a) Find $P(I_2=1) = \int_{x=0}^{x=1} \underbrace{P(I_2=1 \mid X=x)}_{=x} \underbrace{f_X(x)}_{=1}\,dx$

$$= \int_{x=0}^{x=1} x\,dx = \frac{x^2}{2}\Big|_0^1 = \boxed{1/2}$$