Stat 134 lec 37

warmup: 11:00–11:10

$P(Y \in dy) = \int_x P(Y \in dy \mid X = x) f_X(x)\,dx$

$X \sim \text{Unif}(0,1)$

$I_1 \mid X = x,\ I_2 \mid X = x \overset{iid}{\sim} \text{Ber}(x)$

a) Find $P(I_1 = 1) = \int_{x=0}^{x=1} \underbrace{P(I_1 = 1 \mid X = x)}_{x} \underbrace{f_X(x)}_{1}\,dx$

$= \int_0^1 x\,dx = \left.\frac{x^2}{2}\right|_0^1 = \boxed{1/2}$

similarly $P(I_2 = 1) = 1/2$

b) Find $P(I_2 = 1 \mid I_1 = 1)$

$= \dfrac{P(I_2 = 1, I_1 = 1)}{P(I_1 = 1)}$

$P(I_2 = 1, I_1 = 1) = \int_0^1 \underbrace{P(I_2 = 1, I_1 = 1 \mid X = x)}_{P(I_2 = 1 \mid X = x) P(I_1 = 1 \mid X = x) = x^2} \underbrace{f_X(x)}_{1}\,dx$

$= \int_0^1 x^2\,dx = 1/3$

$P(I_2 = 1 \mid I_1 = 1) = \dfrac{1/3}{1/2} = \boxed{2/3}$

Are I_1, I_2 independent?

Last time

Sec 6.3 Conditional densities.

Conditional Prob mass function: $\boxed{P_{Y|X=x}(y) = \frac{P(x,y)}{P(x)}}$

(discrete X, Y)

Conditional density: $\boxed{f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}}$

(continuous X, Y)

Rule of average conditional probabilities (discrete case)

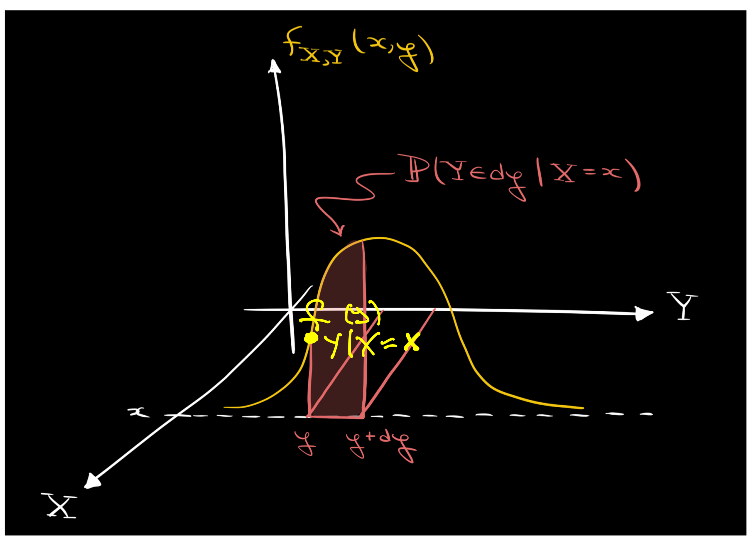
Let X and Y be discrete RV w/ joint distribution $P(X=x, Y=y)$

$$P(Y=y) = \sum_x P(Y=y \mid X=x) P(X=x)$$

Rule of average conditional probabilities (Continuous case)

$$P(Y \in dy) = \int_{x \in X} P(Y \in dy \mid X=x) f_X(x) dx$$

$$= \int_{x \in X} f_{Y|X=x}(y) dy f_X(x) dx$$



The multiplication rule is

$f(x,y) = f_{Y|X=x}(y)\, f_X(x)$

$X \sim \text{Gamma}(r, \lambda)$

$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x > 0$

Ex

let $X \sim \text{Gamma}(2, \lambda)$

$Y|X=x \sim \text{Unif}(0, x)$

a) Find $f_{Y|X=x}(y) = \begin{cases} \frac{1}{x} & \text{for } 0 < y < x < \infty \\ 0 & \text{else} \end{cases}$

b) Find $f(x,y) = \frac{1}{x} \cdot \lambda^2 x e^{-\lambda x} = \boxed{\lambda^2 e^{-\lambda x}, \quad 0 < x < \infty}$

$\frac{1}{x} = f_{Y|X=x}(y)$, $\quad \lambda^2 x e^{-\lambda x} = f_X(x)$

Today <u>Sec 6.3</u>

① <u>Bayesian statistics</u>

② <u>Conjugate Pairs</u>

Sec 6.3

① Bayesian statistics

In frequentist statistics we interpret probability as a long run average constant known only to Tyche, the goddess of fortune.

In Bayesian statistics we interpret probability as a RV

ex When probability a coin lands head is a RV X rather than an unknown constant we are doing Bayesian statistics,

i.e

$$X \sim \text{Unif}(0,1)$$

$$I_1 | X=x,\ I_2 | X=x \overset{iid}{\sim} \text{Ber}(x)$$

CAUTION X is continuous and I_1 is discrete

We write $P(I_1 | X=x)$ for conditional probability mass function (pmf) of I_1 and $f_{X|I_1=1}(x)$ for the conditional density of X

$$P(I_1=1, X=x) \overset{\text{multiplication rule}}{=} P(I_1=1 | X=x) \cdot f_X(x)$$

$$\parallel$$

$$P(X=x, I_1=1) \overset{\text{multiplication rule}}{=} f_{X|I_1=1}(x) \cdot P(I_1=1)$$

$\Rightarrow$ posterior

$$f_{X|I_1=1}(x) = \frac{\overbrace{P(I_1=1 | X=x)}^{\text{likelihood}} \cdot \overbrace{f_X(x)}^{\text{Prior}}}{\underbrace{P(I_1=1)}_{\text{constant}}}$$

Posterior $\propto$ likelihood $\cdot$ Prior

ex Find $f_{X|I_1=1}(x) = \dfrac{x \cdot 1}{1/2} = 2x$

Review Beta Distribution

variable part,

$X \sim \text{Beta}(r,s)$

$$f_X(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}, \quad 0<x<1$$

where $r \in \mathbb{Z}^+ \Rightarrow \Gamma(r) = (r-1)!$

ex If $0<x<1$,

$$f_X(x) \propto 1 \Rightarrow X \sim \text{Beta}(1,1)$$

$$f_X(x) \propto x \Rightarrow X \sim \text{Beta}(2,1)$$

$$f_X(x) \propto x(1-x) \Rightarrow X \sim \text{Beta}(2,2)$$

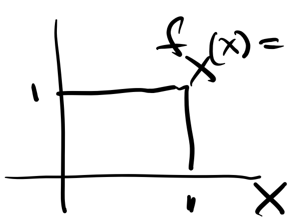
ex $X \sim \text{Unif}(0,1)$

$I_1 \mid X=x,\ I_2 \mid X=x \overset{iid}{\sim} \text{Ber}(x)$

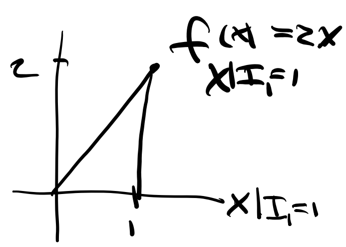
Prior density $f_X(x) = 1 \Rightarrow X \sim \text{Unif}(0,1) = \text{Beta}(1,1)$

Posterior density $f_{X \mid I_1=1}(x) = 2x \Rightarrow X \mid I_1=1 \sim \text{Beta}(2,1)$

Prior $X \sim \text{Unif}(0,1)$



Posterior



ex Let A be an event and

$X \sim \text{Unif}(0,1)$

Suppose $P(A \mid X = x) = x$

Find $f_{X \mid A^c}(x)$

$X \sim \text{Beta}(r,s)$

$f_X(x) = \dfrac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}, \quad 0 < x < 1$

variable part.

$f_{X \mid A^c} \propto \text{likelihood} \cdot \text{prior}$

$\text{likelihood} = P(A^c \mid X = x) = 1 - x$

$\text{prior} = f_X(x) = 1$

$P(A^c \mid X = x) \cdot f_X(x) = 1 - x$ a var part of density

$X \mid A^c \sim \text{Beta}(1,2)$

then $f_{X \mid A^c}(x)$ is density of a $\text{Beta}(1,2)$,

Stat 134

1. Let A, B and C be events and let X be a random variable uniformly distributed on (0,1). Suppose conditional on X=x, that A, B, and C are independent each with probability x. The conditional density of X given that A and B occurs and C doesn't is: i.e $X \mid ABC^c \sim$?

   a $Beta(2,2)$

   b $Beta(3,2)$

   c $Beta(2,3)$

   d none of the above

$f_{X \mid ABC^c} \propto$ likelihood · prior

$$= P(ABC^c \mid X=x)\, f_X(x)$$

$$P(ABC^c \mid X=x) = P(A \mid X=x)\, P(B \mid X=x)\, P(C^c \mid X=x) = x^2(1-x)$$

$\Rightarrow X \mid ABC^c \sim Beta(3,2)$

② Sec 6.3 conjugate pairs

The posterior can be difficult to calculate except when the prior and likelihood are conjugate pairs:

ex prior $X \sim \text{beta}(r,s)$
likelihood $Y \sim \text{Bin}(n,X)$

Posterior $\propto$ likelihood $\cdot$ prior

$$f_{X|Y=j}(x) \propto P(Y=j|X=x)\, f_X(x)$$

similar

$$\underbrace{x^{j}(1-x)^{n-j} \cdot x^{r-1}(1-x)^{s-1}}_{=\; x^{j+r-1}(1-x)^{n-j+s-1}}$$

$$\Rightarrow X|Y=j \sim \text{Beta}(j+r,\, n-j+s)$$

Defn (conjugate pairs)

The prior and likelihood are conjugate pairs when the prior and posterior belong to the same distribution family.