stat 134 lec 38

Warmup 11:00-11:10

rule of average conditional probability:

Suppose $Y|X=x \sim Pois(x)$ where $X \sim Exp(\lambda)$.

Show $Y \sim Geom\left(\frac{\lambda}{\lambda+1}\right)$ on $0,1,2,\dots$

Hint: $P(Y=y) = \int_{x=0}^{\infty} P(Y=y|X=x) f_X(x)dx$

if $Y \sim Geom(p)$ on $0,1,2,\dots$

$P(Y=y) = q^y p$

Here $p = \frac{\lambda}{\lambda+1}$

$q = 1-p = \frac{1}{\lambda+1}$

$$P(Y=y) = \int_0^\infty \frac{e^{-x}x^y}{y!} \cdot \lambda e^{-\lambda x} dx$$

$$= \frac{\lambda}{y!}\int_0^\infty \underbrace{x^y e^{-x} \cdot e^{-\lambda x}}_{\text{variable part of Gamma}(y+1,\lambda+1)} dx \qquad e^{-x}\cdot e^{-\lambda x} = e^{-x(\lambda+1)}$$

$$= \frac{\lambda}{y!} \frac{1}{\text{const part of Gamma}(y+1,\lambda+1)} = \frac{\lambda}{y!}\cdot\frac{1}{\frac{(\lambda+1)^{y+1}}{\Gamma(y+1)}} = \frac{\lambda}{y!}\cdot\frac{y!}{(\lambda+1)^{y+1}}$$

$$= \frac{\lambda}{\cancel{y!}} \frac{\cancel{y!}}{(\lambda+1)^{y+1}} = \left(\frac{1}{\lambda+1}\right)^y \cdot \left(\frac{\lambda}{\lambda+1}\right)$$

← geom formula for Geom$\left(\frac{\lambda}{\lambda+1}\right)$.

Announcement: Q4 Wednesday Dec 1

1) Sec 6.1, 6.2 Conditional prob and expectation in the discrete case

2) Sec 6.3 Average conditional probability and Conjugate pairs

Last time

Sec 6.3 conjugate pairs

When the prior and the posterior belong the same distribution family we say that the prior and the likelihood are conjugate.

ex Prior = Beta and likelihood = Binomial are conjugate pairs

var part of beta $x^{r-1}(1-x)^{s-1}$

Var part of bin $x^{j}(1-x)^{n-j}$

Today

① Sec 6.3 conjugate pairs

② Sec 6.4 covariance and the variance of sum

(1) Sec 6.3 conjugate pairs

eg. Let $Y|X=x \sim \text{Pois}(x)$ where $X \sim \text{Exp}(\lambda)$

likelihood / prior

What distribution is $X|Y=y$?

posterior

Posterior $\propto$ likelihood $\cdot$ prior

$$f_{X|Y=y}(x) \propto \frac{e^{-x} x^y}{y!} \cdot \lambda e^{-\lambda x} = \frac{\lambda}{y!} \underbrace{x^y e^{-x(\lambda+1)}}_{\text{var part of Gamma}(y+1,\, \lambda+1)}$$

$\Rightarrow X|Y=y \sim \text{Gamma}(y+1, \lambda+1)$

Prior = Gamma, likelihood Poisson are conjugate pairs.

note $\text{Exp}(\lambda) = \text{Gamma}(r=1, \lambda)$

$\underline{\underline{ex}}$ Suppose $\Theta \sim \text{Gamma}(r, \lambda)$ with r, λ known.

Let $(N_1|\Theta=\theta, N_2|\Theta=\theta, N_3|\Theta=\theta) \overset{iid}{\sim} \text{Pois}(\theta)$.

Find the posterior distribution of Θ.

Gamma: $f_\Theta(\theta) \propto \theta^{r-1} e^{-\lambda\theta}$

Poisson: $P(N=n|\Theta=\theta) \propto e^{-\theta}\theta^{n}$

$$f_{\Theta|N_1=n_1, N_2=n_2, N_3=n_3} \propto \text{likelihood} \cdot \text{prior}$$

$$\propto \left(e^{-\theta}\theta^{n_1}\right)\left(e^{-\theta}\theta^{n_2}\right)\left(e^{-\theta}\theta^{n_3}\right) \cdot \theta^{r-1} e^{-\lambda\theta}$$

$$= \underbrace{\theta^{(n_1+n_2+n_3+r-1)} e^{-(3+\lambda)\theta}}$$

var part of Gamma $(n_1+n_2+n_3+r, 3+\lambda)$

$\Rightarrow$ prior = Gamma and likelihood Poisson is a conjugate pair.

(2) <u>Sec 6.4</u> <u>Covariance and variance of a sum</u>

$X, Y, \quad S = X + Y$

mean $\mu_X, \mu_Y, \mu_S = \mu_X + \mu_Y$

$\mathrm{Var}(S) = E(\underbrace{(S-\mu_S)}_{= D_S}{}^2)$

$D_S = S - \mu_S$ $\quad$ D_S deviation from mean

$= X + Y - (\mu_X + \mu_Y)$

$= D_X + D_Y$

$\mathrm{Var}(S) = E((D_X + D_Y)^2)$

$= E(D_X^2 + D_Y^2 + 2D_X D_Y)$

$= \underset{\mathrm{Var}(X)}{\underset{\shortparallel}{E(D_X^2)}} + \underset{\mathrm{Var}(Y)}{\underset{\shortparallel}{E(D_Y^2)}} + 2\underbrace{E(D_X D_Y)}_{= \mathrm{Cov}(X,Y)}$

<u>Defn</u> The Covariance of X and Y is

$\mathrm{Cov}(X,Y) = E(\underset{D_X}{(X-\mu_X)}\underset{D_Y}{(Y-\mu_Y)})$

Bilinearity Properties

Proved end of lecture, Appendix

(a) $\text{Cov}(X+Y, Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z)$

(b) $\text{Cov}(aX, bY) = ab\,\text{Cov}(X,Y)$

<u>More generally</u>

$$\text{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \,\text{Cov}(X_i, Y_j)$$

Proved end of lecture,

Thm $\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

<u>Easy facts</u>

$\text{Cov}(X,X) = E(D_X \cdot D_X)$

$\text{Cov}(X,X) = \text{Var}(X)$

$E(D_X D_Y) = \text{Cov}(X,Y) = \text{Cov}(Y,X) = E(D_Y D_X)$

$\text{Cov}(X,c) = 0$ $\quad D_c = c - \mu_c = c - c = 0$

Constant

ex

Simplify

$$\mathrm{Cov}(X-5Y,\ 3X+Y-Z+10)$$

$$= 3\,\mathrm{Var}(X) + \mathrm{Cov}(X,Y) - \mathrm{Cov}(X,Z) + 0$$

$$-15\,\mathrm{Cov}(X,Y) - 5\,\mathrm{Var}(Y) + 5\,\mathrm{Cov}(Y,Z) + 0$$

Recall X, Y independent

$$\Rightarrow \quad E(XY) = E(X)E(Y)$$

$\mathrm{Cov}(X,Y) = 0$ if X, Y independent.

Hence if X, Y indep,

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\overset{=0}{\mathrm{Cov}(X,Y)}$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y)$$

Stat 134

Wednesday April 24 2019

1. Consider a Poisson(λ) process. Let $T_r \sim$ gamma(r, λ) be the rth arrival time. Cov(T_1, T_3) equals:

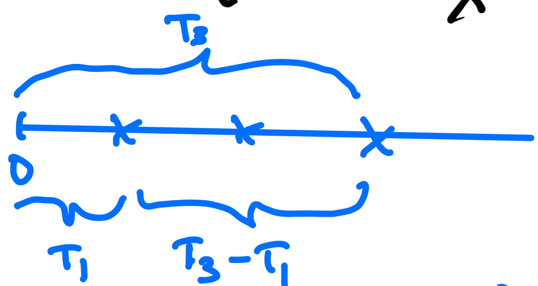
a λ

b λ^2

c $1/\lambda^2$

d none of the above

Recall $\mathrm{Var}(T_r) = \dfrac{r}{\lambda^2}$



$$\underbrace{\mathrm{Var}(T_3 - T_1)}_{=\,T_2} = \underbrace{\mathrm{Var}(T_3)}_{=\,\frac{3}{\lambda^2}} + \underbrace{\mathrm{Var}(T_1)}_{=\,\frac{1}{\lambda^2}} - 2\,\mathrm{Cov}(T_3, T_1)$$

$$\mathrm{Var}(T_2) = \frac{2}{\lambda^2}$$

$$= -\frac{2}{\lambda^2} = -2\,\mathrm{Cov}(T_3, T_1)$$

$$\Rightarrow \boxed{\mathrm{Cov}(T_3, T_1) = \frac{1}{\lambda^2}}$$

Appendix

Bilinearity Properties

Thm

ⓐ $\text{Cov}(X+Y, Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z)$

ⓑ $\text{Cov}(aX, bY) = ab\,\text{Cov}(X,Y)$

Pf/

a)

$$\text{Cov}(X+Y, Z) = E\big((X+Y-\mu_{X+Y})(Z-\mu_Z)\big)$$

$$= E\big([(X-\mu_X)+(Y-\mu_Y)](Z-\mu_Z)\big)$$

$$= E\big((X-\mu_X)(Z-\mu_Z) + (Y-\mu_Y)(Z-\mu_Z)\big)$$

$$= E\big((X-\mu_X)(Z-\mu_Z)\big) + E\big((Y-\mu_Y)(Z-\mu_Z)\big)$$

$$= \text{Cov}(X,Z) + \text{Cov}(Y,Z). \quad \square$$

b)

$$\text{Cov}(aX, bY) = E\big((aX-\mu_{aX})(bY-\mu_{bY})\big)$$

$$= E\big((aX-a\mu_X)(bY-b\mu_Y)\big)$$

$$= E\big(ab(X-\mu_X)(Y-\mu_Y)\big)$$

$$= ab\,E\big((X-\mu_X)(Y-\mu_Y)\big)$$

$$= ab\,\text{Cov}(X,Y) \quad \square$$

Appendix

Thm $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

Pf/ $\mathrm{Cov}(X,Y) = E(D_X D_Y) = E\big((X-\mu_X)(Y-\mu_Y)\big)$

$$= E(XY - \mu_X Y - X\mu_Y + \mu_X\mu_Y)$$

$$= E(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y)$$

$$= E[XY) - E[X]E(Y)$$

$\square$