Suppose you draw a number from a bag, with equal probabilities across the choices \( \{1, 2, 3\} \).

Once you draw a number, you toss a coin until you get that many number of heads followed by a tails—so if you draw a 3, you keep tossing until you encounter the sequence \{Heads, Heads, Heads, Tails\}.

What is the probability of tossing a coin seven times given that you draw the number 2?

Find \( P(\text{7|draw 2}) \)

\[
\begin{align*}
&\text{HHHT} \\
&\text{HHHT} \\
&\text{HHHT} \\
&\text{HHHT} \\
&\text{HHHT} \\
&\text{HHHT} \\
&\text{HHHT} \\
&\text{HHHT} \\
&\text{HHHT} \text{ or HHHT} \\
&\text{so } 12 \text{ possibilities} \\
&\text{each with probability } \left(\frac{1}{2}\right)^7
\end{align*}
\]

\[
= 12 \left(\frac{1}{2}\right)^7
\]
Last time

sec 1.4 Independence

Note that if \( P(AB) = P(A)P(B) \) then \( P(AB) = P(A^c)P(B^c) \)

Since,

\[
P(AB) = P((AB)^c B) = P(B) - P(AB)
\]

For independence of \( AB \)

\[
P(B) - P(A|B) = (1 - P(A))P(B) = P(A^c)P(B)
\]

If \( A \) and \( B \) are independent then so too is \( A, B^c \), and \( A^c, B \) and \( A^c, B^c \).

sec 1.5 Bayes rule

There are two types of conditional probabilities:

\( P(A|B) \) is forward conditional (likelihood conditional)

\( P(B|A) \) is backwards conditional (posterior conditional)

Today

1. sec 1.6 independence of 3 or more events
2. sec 2.1 Binomial Distribution
### Section 6. Independence of 3 Events

**Defn (pairwise independence of 3 events)**

A, B, C are pairwise independent if

\[ P(AB) = P(A)P(B) \quad \text{and} \quad P(AC) = P(A)P(C) \quad \text{and} \quad P(BC) = P(B)P(C) \]

---

One ball is drawn randomly from a bowl containing four balls numbered 1, 2, 3, and 4. Define the following three events:

- Let A be the event that a 1 or 2 is drawn. That is, \( A = \{1, 2\} \).
- Let B be the event that a 1 or 3 is drawn. That is, \( B = \{1, 3\} \).
- Let C be the event that a 1 or 4 is drawn. That is, \( C = \{1, 4\} \).

Is A, B, C pairwise independent?

\[ P(AB) = \frac{1}{4} \neq P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} \quad \text{(No)} \]

Similarly,

\[ P(AC) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} \quad \text{(Yes)} \]

\[ P(BC) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} \quad \text{(Yes)} \]

Is \( P(ABC) = P(A)P(B)P(C) \)

\[ P(ABC) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \quad \text{(No)} \]

---

**Defn (mutual independence of 3 events)**

A, B, C are mutually independent if

\[ P(ABC) = P(A)P(B)P(C), \quad \text{(and the same for any of the events replaced by its complement)} \]
we require showing 8 equations is true for mutual independence. This is a strong condition.

Thus suppose A, B, C are mutually independent. Then they are also pairwise independent.

Proof:

We can write

\[ P(AB) = P(ABC) + P(ABC^c) \]

"add" rule

\[ = P(A)P(B)P(C) + P(A)P(B)P(C^c) \]

\[ = P(A)P(B)[P(C) + P(C^c)] \]

\[ = P(A)P(B). \quad \text{ "1} \]

Similarly for other cases.
Note that \( P(ABC) = P(A)P(B)P(C) \) by itself does not imply pairwise independence:

Let \( \Omega = \{1, 2, 3, 4, 5, 6, 7, 8\} \)

\( A = B = \{1, 2, 3, 4\} \)

\( C = \{1, 5, 6, 7\} \)

Is \( P(ABC) = P(A)P(B)P(C) \)?

\( P(A) = P(B) = P(C) = \frac{1}{2} \)

\( P(ABC) = P(\{1\}) = \frac{1}{8} \)

\( \Rightarrow P(ABC) = P(A)P(B)P(C) \checkmark \)

Is \( A, B, C \) pairwise independent?

**No** \( P(AB) \neq P(A)P(B) \)

\( \frac{1}{2} \neq \frac{1}{2} \frac{1}{2} \frac{1}{2} \)

Thus, \( A, B, C \) are mutually independent iff

1) \( A, B, C \) are pairwise independent
2) \( P(ABC) = P(A)P(B)P(C) \).
Sketch

Suppose (1) and (2) hold,

let's show \( P(ABC^c) = P(A)P(B)P(C^c) \)

write

\[
P(ABC^c) = \frac{P(ABC^c) + P(ABC) - P(ABC)}{P(AB)}
\]

\[
\Rightarrow P(ABC^c) = P(A)P(B) - P(A)P(D)P(C)
\]

\[
= P(A)P(B)[1 - P(C)]
\]

\[
= P(A)P(B)P(C^c)
\]

Similarly for other cases.
sec 2.1 Binomial distribution

Bernoulli (p) trial (distribution)

- two outcomes
  - success
  - failure

Ex: roll a die.
Success is getting a 6
Failure is not getting a 6

Binomial (n, p) distribution (Bin (n, p))

we have n independent Bernoulli (p) trials

Ex: we roll a die n times,
What are the possible number of successes?

In Bin(n, p) the chance of having k successes (0 ≤ k ≤ n) is given by the Binomial formula:

\[ P(k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \]

Ex: You roll a die 5 times. What is the chance of getting 2 sixes?

\[ n = 5, \quad k = 2, \quad p = \frac{1}{6} \]
\[ P(2) = \frac{5!}{2!3!} \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^3 = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^3 = \frac{5!}{2!3!} \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^3 \]

What is chance of getting:

\begin{align*}
\text{success (6)} &: 1 \ \text{0} \ 0 \ 0 \ 0 \ ? \\
\text{failure (not 6)} &: 0 \ 1 \ 1 \ 0 \ 0 \ ? \\
\vdots & \cdots \\
\text{How many of these are there?} & \begin{array}{c}
\frac{5!}{2!3!} \\
\end{array}
\end{align*}

We write \( \frac{5!}{2!3!} \approx \binom{5}{2} \approx \binom{5}{3} \approx \binom{5}{2,3} \)
A well shuffled deck is cut in half so there are 7 aces in the first half deck and 6 aces in the second half deck. Five cards are dealt off the top of one half deck and five cards are dealt off the top of the other half deck. The binominal formula doesn’t apply to find the chance of getting exactly three diamonds total because:

a. The probability of a trial being successful changes

b. The trials aren’t independent

c. There isn’t a fixed number of trials

d. more than one of the above