Warmup 11:00 – 11:10

Let (X,Y) be bivariate normal. Then (2X+3Y+4, 6X-Y-4) is bivariate normal.

a true

b false

c not enough info to decide

Show $a(2X+3Y+4)+b(6X-Y-4)$ is normal for all a,b.

$$= \underbrace{(2a+6b)X+(3a-b)Y}_{\text{normal since } (X,Y) \text{ BVN}} + \underbrace{4a-4b}_{\text{constant}} \text{ is normal}$$

$\Rightarrow (2X+3Y+4,\ 6X-Y-4)$ is BVN

Announcement:
Review materials are on stat134.org website.

Total Variance decomposition — Sec 6.4

Properties (from lec 34)

1. $E(Y) = E(E(Y|X))$ iterated expectation
2. $E(aY+b|X) = aE(Y|X)+b$
3. $E(Y+Z|X) = E(Y|X)+E(Z|X)$
4. $E(g(X)|X) = g(X)$
5. $E(g(X)Y|X) = g(X)E(Y|X)$
6. $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$ total variance decomposition (sec. 6.2.18)

Lab 1	Lab 2	Lab 3	Lab 4	Lab 5	Lab 6	Lab 7
4.13	3.86	4.00	3.88	4.02	4.02	4.00
4.07	3.85	4.02	3.88	3.95	3.86	4.02
4.04	4.08	4.01	3.91	4.02	3.96	4.03
4.07	4.11	4.01	3.95	3.89	3.97	4.04
4.05	4.08	4.04	3.92	3.91	4.00	4.10
4.04	4.01	3.99	3.97	4.01	3.82	3.81
4.02	4.02	4.03	3.92	3.89	3.98	3.91
4.06	4.04	3.97	3.90	3.89	3.99	3.96
4.10	3.97	3.98	3.97	3.99	4.02	4.05
4.04	3.95	3.98	3.90	4.00	3.93	4.06

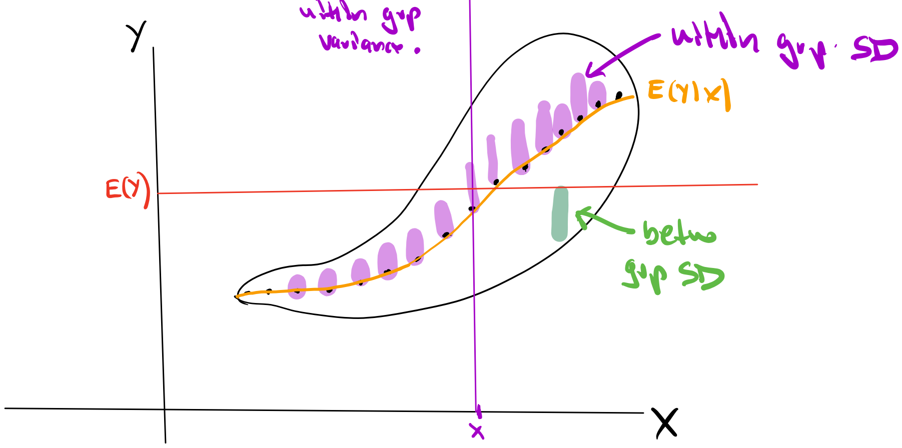
total variance is the average of the group variances (written $E(Var(Y|X))$) plus the variance of the group averages (written $Var(E(Y|X))$.

This is proven in exercise 6.2.18.

Picture

tot var = avg($\;^2$) + $\;^2$ — var of $E(Y|X)$ (between group variance)

within grp variance.



ex Let $U \sim \text{Unif}(0,1)$ and $X|U \sim \text{Exp}\left(\frac{1}{U}\right)$

Recall if $E\left(\text{Exp}\left(\frac{1}{U}\right)\right) = U \Rightarrow E(X|U) = U$

$Var\left(\text{Exp}\left(\frac{1}{U}\right)\right) = U^2 \Rightarrow Var(X|U) = U^2$

Find $E(X)$, $E(UX)$ and $Var(X)$.

$E(X) = E(E(X|U)) = E(U) = 1/2$

$E(UX) = E(E(UX|U)) = E(U \underbrace{E(X|U)}_{= U})$

$= E(U^2) = \underbrace{Var(U)}_{= 1/12} + \underbrace{E(U)^2}_{= (1/2)^2} = 1/3$

$Var(X) = \underbrace{E(\underbrace{Var(X|U)}_{= U^2})}_{= 1/3} + \underbrace{Var(\underbrace{E(X|U)}_{= U})}_{= 1/12} = 5/12$

Joy Lin

1:11pm

HW 13, 6.3.12, b)

6.rev.21, c)

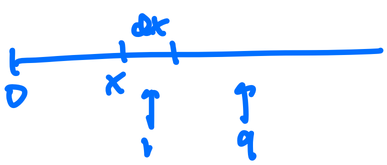
12. Suppose there are ten atoms, each of which decays by emission of an α-particle after an exponentially distributed lifetime with rate 1, independently of the others. Let T_1 be the time of the first α-particle emission, T_2 the time of the second. Find:

a) the distribution of T_1;

b) the conditional distribution of T_2 given T_1;

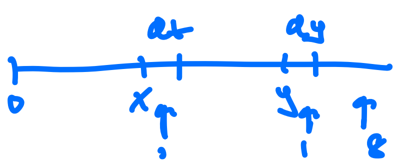
c) the distribution of T_2.

a) $T_1, \dots, T_{10} \overset{iid}{\sim} \exp(1)$



$$P(T_{(1)} \in dx) = \binom{10}{1,9} 1 \cdot e^{-1\cdot x} \cdot (e^{-x})^9 \, dx = \boxed{10 e^{-10x} \, dx}$$

b)



$$P(T_{(1)} \in dx, T_{(2)} \in dy) = \binom{10}{1,1,8} 1 e^{-x} \, 1 \cdot e^{-y} (e^{-y})^8$$
$$= 10 \cdot 9 \, e^{-x} e^{-9y}$$

$$\Rightarrow P(T_{(2)} \in dy \mid T_{(1)} \in dx) = \frac{P(T_{(1)} \in dx, T_{(2)} \in dy)}{P(T_{(1)} \in dx)}$$
$$= \frac{90 e^{-x} e^{-9y}}{10 e^{-10x}} \frac{\cancel{dx}\, dy}{\cancel{dx}} = \boxed{9 e^{-9(y-x)} \, dy}$$

c) $P(T_{(2)} \in dy) = \int_{x=0}^{y} 9e^{-9(y-x)} 10e^{-10x} dx dy$

$= 90e^{-9y} dy \int_{x=0}^{y} e^{-x} dx = \boxed{90e^{-9y}(1-e^{-y}) dy}$

21. I toss a coin which lands heads with probability p. Let W_H be the number of tosses till I get a head, W_{HH} the number of tosses till I get two heads in a row, and W_{HHH} the number of tosses till I get three heads in a row. Find:

a) $E(W_H)$; b) $E(W_{HH})$ [*Hint*: condition on whether the first toss was heads or tails]; c) $E(W_{HHH})$ [*Hint*: condition on W_T].

d) Generalize to find the expected number of tosses to obtain m heads in a row.

better to condition on W_{HH}

$E(W_{HHH}) = E(E(W_{HHH} | W_{HH}))$

$= \sum_{s=1}^{\infty} E(W_{HHH} | W_{HH}) \cdot P(W_{HH} = s)$

let $\mu = E(W_{HHH})$

W_{HH}	$P(W_{HH})$	$E(W_{HHH} \mid W_{HH})$	
2	p^2	$(\mu+3)q + 3p$	HHT, HHH
3	qp^2	$(\mu+4)q + 4p$	THHT, THHH
4	q^2p^2	$(\mu+5)q + 5p$	TTHHT, TTHHH
5	$\vdots$		
$\vdots$			

$$\mu = \sum_{x=2}^{\infty}\left[(x+1+\mu)q + (x+1)p\right] P(W_{HH}=x)$$

$$= \sum_{x=2}^{\infty}\left[(x+1) + q\mu\right] P(W_{HH}=x)$$

$$= E(W_{HH}+1) + \mu q$$

$$= \frac{1}{p} + \frac{1}{p^2} + 1 + \mu q$$

$$\Rightarrow p\mu = 1 + \frac{1}{p} + \frac{1}{p^2}$$

$$\Rightarrow \boxed{\mu = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3}}$$

Ksenia Bogdanova

6:36pm

HW 11, 5.rev.24 (c and d)

24. A coin of diameter d is tossed at random on a grid of squares of side s. Making appropriate assumptions, to be stated clearly, calculate:

a) the probability that the coin lands inside some square (i.e., not touching any line);

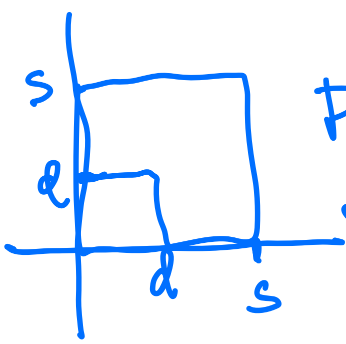
b) the probability that the coin lands heads inside some square.

Suppose now that the coin is tossed four times. Let X be the number of times it lands inside a square, Y the number of heads. Assume $d = s/2$. Calculate:

c) $P(X = Y)$; d) $P(X < Y)$; e) $P(X > Y)$.

a) assume $d < s$

No difference if square coin side d.



$P(\text{coin in square})$

$= P(0 < \text{right edge} < s-d,\ 0 < \text{bot edge} < s-d)$

$= \left(\frac{s-d}{s}\right)^2$

b) assume getting heads indep of position of coin. Assume fair coin.

$P(\text{coin in square, heads})$

$= \frac{1}{2}\left(\frac{s-d}{s}\right)^2$

c) $d = 5/2$

X = # times (out of 4) coin lands in square

$$P(\text{coin in square}) = \left(\frac{5/2}{5}\right)^2 = 1/4$$

$$X \sim Bin(4, 1/4)$$

Y = # heads (out of 4)

$$Y \sim Bin(4, 1/2)$$

$$P(X=Y) = P(X=0, Y=0) + P(X=1, Y=1)$$

$$+ P(X=2, Y=2) + P(X=3, Y=3) + P(X=4, Y=4)$$

$$= (3/4)^4 \cdot (1/2)^4 + 4^2 \tfrac{1}{4} (3/4)^3 (1/2)^4 + 6^2 (1/4)^2 (3/4)^2 (1/2)^4$$

$$+ 4^2 \left(\tfrac{1}{4}\right)^3 (3/4) (1/2)^4 + (1/4)^4 (1/2)^4$$

d) $P(X<Y)$

$= P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3) + P(X=0, Y=4)$

$+ P(X=1, Y=2) + P(X=1, Y=3) + P(X=1, Y=4)$

$+ P(X=2, Y=3) + P(X=2, Y=4) + P(X=3, Y=4)$

This is straight forward to find.