Stat 134 lec 8

warmup

Find the probability that a poker hand has two 2 of a kind

eg K, K, Q, Q, 7

$$\frac{\overset{\text{single}}{\binom{13}{1}}\overset{\text{double}}{\binom{12}{2}}\overset{K}{\binom{4}{2}}\overset{Q}{\binom{4}{2}}\overset{7}{\binom{4}{1}}}{\binom{52}{5}} = \frac{\overset{\text{double}}{\binom{13}{2}}\overset{\text{single}}{\binom{11}{1}}\overset{K}{\binom{4}{2}}\overset{Q}{\binom{4}{2}}\overset{7}{\binom{4}{1}}}{\binom{52}{5}}$$

Note $\binom{13}{1}\binom{12}{2} = \frac{13}{1}\cdot\frac{12\cdot 11}{2}$

$\binom{13}{2}\binom{11}{1} = \frac{13\cdot 12}{2}\cdot\frac{11}{1}$

aabbc = bbaac ← since b is a double on both left and right

aabbc ≠ aaccb ← since on the left b is a double and on the right b is a single

Find the probability that an 6 card poker hand has two 2 of a kind and 2 single

eg K, K Q Q 7, 8

$$\frac{\binom{13}{2}\binom{11}{2}\binom{4}{2}\binom{4}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{6}}$$

$\binom{52}{6}$

Last time

independent trials (draw w/ replacement) — binomial distribution — 2 outcome trial
— multinomial distribution — K outcome trial

dependent trials (draw w/o replacement) — hypergeometric distribution — 2 outcome trial
— multivariate hypergeometric distribution — K outcome trial

Sec 2.5 hypergeometric distribution

abbrev. $HG(n, N, G)$ Parameters: N = population size
G = # Good in population
n = sample size.

Suppose a population of size N contains G good and B bad elements $(N = G + B)$.
A sample, size n, with g good and b bad elements $(n = g + b)$ is chosen at random without replacement.

$$P(g \text{ good}) = \frac{\binom{G}{g}\binom{B}{b}}{\binom{N}{n}}$$

Stat 134

1. The probability of being dealt a three of a kind poker hand (ranks $aaabc$ where $a \neq b \neq c$) is:

a $\binom{4}{3}\binom{4}{1}\binom{44}{1}/\binom{52}{5}$

b $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$

c $\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{1}\binom{44}{1}/\binom{52}{5}$

d none of the above

c	First 13 choose 1 to designate the rank of the three of a kind, then 4 choose 3 to get the 3 of a kind, then 12 choose 1 to designate the 2nd rank and 4 choose 1 to get 1 card of that kind, and finally pick 1 from the rest 44 cards
b	Choose a rank out of 13, then choose 3 cards out of that rank, then choose 2 ranks out of the rest 12, each pick 1 card

today → (1) sec 2.5 Binomial approx to hypergeometric.

(2) sec 3.1 – random variables (RV)

joint distribution of 2 RVs and independence

① Sec 2.5 Binomial approx to hypergeometric.

Binomial — independent trials
Hypergeometric — dependent trials.

ex 100 person class with a grade distribution:

A grade : 70 students
B grade : 30 students.

Sample 5 students at random w/o replacement (SRS).

Find $P(3A\text{'s}, 2B\text{'s})$

exact
hypergeometric
$$= \frac{\binom{70}{3}\binom{30}{2}}{\binom{100}{5}} = \binom{5}{3}\frac{70}{100}\frac{69}{99}\frac{68}{98}\frac{30}{97}\frac{29}{96} = .316$$

approx
binomial
$$= \binom{5}{3}(.7)^3(.3)^2 = .309$$

when N is large relative to n, $HG(5, 100, 70) \approx Bin(5, .7)$

why?

$$HG(n, N, G) \approx Bin\left(n, \frac{G}{N}\right)$$

Summary of approximations

$HG(n, N, G)$

approx by binomial
N large, n small
$p = \frac{G}{N}$

binomial (n, p)

approx by Poisson
$p \to 0,\ n \to \infty,\ np \to \mu$

Poisson (μ)

approx by normal
n large
$\mu = np,\ \sigma = \sqrt{npq}$
$0 < \mu \pm 3\sigma < n$
use continuity correction

Normal (μ, σ^2)

2 Sec 3.1 Intro to Random Variables (RV)

A RV, X, is the outcome of an experiment.

What distribution is the following RV?

X = The number of aces in 5 cards drawn from a standard deck?

$X \sim HG(5, 52, 4)$

ex flip a prob p coin 2 times

X = # heads

we write $X \sim Bin(2, p)$

More precisely,

$$X : \underbrace{\Omega}_{\text{outcome space}} \longrightarrow \mathbb{R} \quad \text{is a function}$$

$$\begin{aligned} HH &\longmapsto 2 \\ HT &\longmapsto 1 \\ TH &\longmapsto 1 \\ TT &\longmapsto 0 \end{aligned}$$

so $X=1$ means $\{HT, TH\} \subseteq \Omega$

$X=1$ is an event

$P(X=1) = \binom{2}{1} p^1 (1-p)^1$ binomial formula

Joint Distribution

Let (X,Y) be the joint outcome of 2 RVs X, Y.

ex

X: one draw from $\boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{3}$

Given $X=x$, Y = number of heads in x coin tosses.

$$P(X=x, Y=y) = P(Y=y \mid X=x)\cdot P(X=x)$$

$$P(X=1, Y=1) = \underbrace{P(Y=1 \mid X=1)}_{=1/2}\cdot \underbrace{P(X=1)}_{=1/4} = 1/8$$

Find, what the range of values of X? 1, 2, 3

Y? 0, 1, 2, 3

$$P(1,0) = \underbrace{P(Y=0 \mid X=1)}_{=1/2}\,\underbrace{P(X=1)}_{=1/4} = 1/8$$

$$P(A,B) = P(A|B)P(B)$$
$$= P(A \cap B)$$

marginal prob of X
$$P(x) = \sum_{y \in Y} P(x,y)$$

$Y \backslash X$	1	2	3	
	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	
3	0	0	$\frac{1}{32} = \frac{1}{8}\cdot\frac{1}{4}$	$\frac{1}{32}$
2	0	$\frac{1}{8} = \frac{1}{4}\cdot\frac{1}{2}$	$\frac{3}{32} = \frac{3}{8}\cdot\frac{1}{4}$	$\frac{7}{32}$
1	$\frac{1}{8} = \frac{1}{2}\cdot\frac{1}{4}$	$\frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2}$	$\frac{3}{32} = \frac{3}{8}\cdot\frac{1}{4}$	$\frac{15}{32}$
0	$\frac{1}{8} = \frac{1}{2}\cdot\frac{1}{4}$	$\frac{1}{8} = \frac{1}{4}\cdot\frac{1}{2}$	$\frac{1}{32} = \frac{1}{8}\cdot\frac{1}{4}$	$\frac{9}{32}$

marginal prob of Y
$$P(y) = \sum_{x \in X} P(x,y)$$

Is X, Y dependent? — Yes

$$\underbrace{P(X=1, Y=3)}_{=0} \neq \underbrace{P(X=1)}_{=\frac{1}{4}}\underbrace{P(Y=3)}_{=\frac{1}{32}}$$

Def^n Two RVs are independent if

$P(Y=y \mid X=x) = P(Y=y)$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$

By the multiplication rule, if X, Y are indep,

$$P(X=x, Y=y) = \underbrace{P(Y=y \mid X=x)}_{= P(Y=y)} P(X=x)$$

so $\boxed{P(X=x, Y=y) = P(X=x)P(Y=y)}$.

Equivalently,

$P(X=x \mid Y=y) = P(X=x)$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

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The joint distribution of X and Y is drawn below:

Y \ X	0	1	2	P(y)
1	1/4	1/3	1/12	2/3
0	1/8	1/6	1/24	1/3
P(x)	3/8	1/2	1/8	

?

$\frac{1}{4} = \frac{3}{8} \cdot \frac{2}{3}$ ✓

$\frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3}$ ✓

a) X and Y are independent ✓

b) If we divide both rows by their marginal probability we get the same answer. ✓

c) $P(X = x|Y = 0) = P(X = x|Y = 1)$ ✓

d) All of the above

b) For b when we divide the row by the marginal we get $\frac{P(x,y)}{P(y)} = \frac{P(x|y)\cancel{P(y)}}{\cancel{P(y)}} = P(x|y)$

$P(x=0\mid y=1)$	$P(x=1\mid y=1)$	$P(x=2\mid y=1)$
$P(x=0\mid y=0)$	$P(x=1\mid y=0)$	$P(x=2\mid y=0)$

=

3/8	1/2	1/8
3/8	1/2	1/8

extra practice

You and a friend are playing poker. If each of you are dealt 5 cards from the same deck, what is the chance that you both get a 4 of a kind? (ranks aaaab, $a \neq b$)

$$\underbrace{P(\text{you get 4 of kind} \mid \text{friend 4 of kind})}_{\dfrac{\binom{11}{1}\binom{4}{4}\left[\binom{10}{1}\binom{4}{1} + \binom{3}{1}\right]}{\binom{47}{5}}} \cdot \underbrace{P(\text{friend gets 4 of kind})}_{= \dfrac{\binom{13}{1}\binom{12}{1}\binom{4}{4}\binom{4}{1}}{\binom{52}{5}}}$$

$$\boxed{\frac{\binom{11}{1}\binom{4}{4}\left[\binom{10}{1}\binom{4}{1} + \binom{3}{1}\right]}{\binom{47}{5}} \cdot \frac{\binom{13}{1}\binom{12}{1}\binom{4}{4}\binom{4}{1}}{\binom{52}{5}}}$$