SOLUTIONS

SID: __________________________

1. (10 pts total) There are 30 marbles in a bag. 10 of them are black, 10 of them are white, and 10 of them are red. Let X denote the number of different colors appearing among 5 marbles selected at random from the bag. Find:

   (a) (3 pts) $P(X = 2)$;

$$\mathbb{P}(X=2) = \binom{3}{2}\mathbb{P}(\text{red \& blue both appear})$$

$$= \binom{3}{2}\left(1 - \mathbb{P}(\text{no red} \cup \text{no blue})\right)$$

$$= 3\cdot\left(1 - 2\cdot\left(\frac{\binom{20}{5}}{\binom{30}{5}}\right) + \frac{\binom{10}{5}}{\binom{30}{5}}\right)$$

For instance: ↑ all B or W ↑ all W

   (b) (3 pts) $\mathbb{E}(X)$;

Let $\mathbb{I}_i$ be the indicator that color i appears in the sample. So $X = \mathbb{I}_b + \mathbb{I}_r + \mathbb{I}_w$. By symmetry,

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{I}_b + \mathbb{I}_r + \mathbb{I}_w) = 3\mathbb{E}(\mathbb{I}_b)$$

$$= 3\left(1 - \frac{\binom{20}{5}}{\binom{30}{5}}\right)$$

   (c) (4 pts) $Var(X)$.

$$Var(X) = \mathbb{E}(X^2) - \underbrace{\mathbb{E}(X)}_{\text{see above.}}{}^2,$$

where

$$\mathbb{E}(X^2) = \mathbb{E}\left((\mathbb{I}_r + \mathbb{I}_b + \mathbb{I}_w)^2\right)$$

$$= 3\mathbb{E}(\mathbb{I}_r^2) + 3\cdot 2\,\mathbb{E}(\overbrace{\mathbb{I}_r\mathbb{I}_b}^{r,\,b \text{ both appear.}})$$

$$= \mathbb{E}(X) + 6\cdot\left(1 - \mathbb{P}(\text{no red} \cup \text{no blue})\right)$$

$$= \mathbb{E}(X) + 6\left(1 - \left(\frac{\binom{20}{5}}{\binom{30}{5}}\right)\cdot 2 + \left(\frac{\binom{10}{5}}{\binom{20}{5}}\right)\right)$$

2. (5 pts) You are given a fair coin and a coin which lands heads with probability $\frac{1}{3}$. Unsure which coin is which, you select one of the coins and decided to toss it until you observe 4 heads; this takes 10 tosses. Given this information, what is the chance you selected the fair coin?

Let A be the event the fair coin was chosen; X trial on which 4^{th} H appears.

Bayes' Rule:

$$\mathbb{P}(A \mid X=10) = \frac{\mathbb{P}(A) \cdot \mathbb{P}(X=10 \mid A)}{\mathbb{P}(X=10)}$$

$$= \frac{\frac{1}{2}\binom{9}{3}\left(\frac{1}{2}\right)^{10}}{\frac{1}{2}\binom{9}{3}\left(\frac{1}{2}\right)^{10} + \frac{1}{2}\binom{9}{3}\left(\frac{1}{3}\right)^{4}\left(\frac{2}{3}\right)^{6}}$$

3. (5 pts) Suppose that bundles of yarn are 60 meters long on average, with an SD of 5 meters, and that a large blanket requires at least 1000 meters of yarn. A thrifty knitter wants to purchase as few bundles as possible. Find the minimum number of bundles they must purchase for there to be at least a 95% chance that they are able to knit the full blanket.

Let $X_n :=$ length of n bundles. So by CLT,

$$X_n \overset{\text{approx.}}{\sim} \mathcal{N}(n \cdot 60, \sqrt{25n}).$$

$$\mathbb{P}(X_n \geq 1000) \approx 1 - \Phi\left(\frac{1000-60n}{\sqrt{25n}}\right) \geq 0.95$$

$$\Leftrightarrow \Phi\left(\frac{1000-60n}{\sqrt{25n}}\right) \leq 0.05$$

$$\Leftrightarrow \frac{1000-60n}{5\sqrt{n}} \leq \Phi^{-1}(0.05)$$

$$\to \frac{1000-60n}{\Phi^{-1}(0.05)} \leq 5\sqrt{n}$$

$$\Leftrightarrow \frac{60}{\Phi^{-1}(0.05)} n + 5\sqrt{n} \leq \frac{1000}{\Phi^{-1}(0.05)}$$

Must complete square; tedious process. Have students set up polynomial equation instead?

4. (5 pts) Three couples attend a dinner. Each of the six people chooses a seat randomly from a round table with six seats. What is the probability that no couple sits together?

Let A_i be the event that couple i sits together.

So $\mathbb{P}(A_i) = \dfrac{2! \times 4! \cdot 6}{6!}$,

$\mathbb{P}(A_i A_j) = \dfrac{6 \cdot 2! \cdot 3 \cdot 2! \cdot 2!}{6!}$, $i \neq j$,

$\mathbb{P}(A_1 A_2 A_3) = \dfrac{6 \cdot 2!^3 \cdot 3!}{6!}$

By inclusion-exclusion, $\mathbb{P}(\text{no couple sits together}) = 1 - \mathbb{P}(\bigcup_{i=1}^{3} A_i)$

$= 1 - 3\mathbb{P}(A_i) + \binom{3}{2}\mathbb{P}(A_i A_j) - \mathbb{P}(A_1 A_2 A_3)$

5. (5 pts) Suppose that on average, 2 moths per 12-hour night are killed by a particular hanging bug zapper. Assume that conditions are the same across different nights and different times of the night, and that moths arrive independently of one another. Find the chance that more than 7 moths are killed in a period of three nights.

Moth deaths follow a Poisson process, with intensity $\frac{2}{12}$ per hour.

So over $3 \cdot 12 = 36$ hours, deaths follow Pois(6) dist'n.

Let X represent this variable.

So $\mathbb{P}(X > 7) = 1 - \mathbb{P}(X \leq 7) = 1 - \sum_{k=0}^{7} e^{-6} \dfrac{6^k}{k!}$