STAT 134: Section 6.5

Adam Lucas

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Because we do not hold discussion sections on days when quizzes are held, we would like to offer a short, supplementary problem. A solution is on the next page.

Problem

Suppose $X \sim \text{Poisson}(1)$ and $Y = 1 + X$.

a. Apply Markov's inequality to bound $\mathbb{P}(X \geq 2)$ and $\mathbb{P}(Y \geq 3)$.

b. $\mathbb{P}(X \geq 2) = \mathbb{P}(Y \geq 3)$, right? Why are the bounds different? What could be done to ensure the bounds from Markov's inequality are the same? Discuss.

Solution

a. Apply Markov's inequality to bound $\mathbb{P}(X \geq 2)$ and $\mathbb{P}(Y \geq 3)$.

Because $\mathbb{E}X = 1$ and $\mathbb{E}Y = 2$, Markov's inequality gives

$$\mathbb{P}(X \geq 2) \leq \frac{1}{2} \quad \text{and} \quad \mathbb{P}(Y \geq 3) \leq \frac{2}{3}.$$

b. $\mathbb{P}(X \geq 2) = \mathbb{P}(Y \geq 3)$, right? Why are the bounds different? What could be done to ensure the bounds from Markov's inequality are the same? Discuss.

Yes, $\mathbb{P}(X \geq 2) = \mathbb{P}(Y \geq 3)$ because

$$\mathbb{P}(X \geq 2) = \mathbb{P}(1 + X \geq 1 + 2) = \mathbb{P}(Y \geq 3).$$

Note that the minimum of X is zero, while—due to the shift of X which defines Y—the minimum of Y is one. Markov's inequality applies to all nonnegative random variables, but we could similarly derive a variant of Markov's inequality which applies to random variables taking values of at least one by first subtracting one and then applying the usual Markov's inequality. For example, if Y is always at least one, then

$$\begin{aligned}(k-1)^{-1}\mathbb{E}Y &= (k-1)^{-1}(1 + \mathbb{E}[Y-1]) \\ &\geq (k-1)^{-1} + \mathbb{P}(Y - 1 \geq k - 1) \\ &= (k-1)^{-1} + \mathbb{P}(Y \geq k).\end{aligned}$$

That is, for $k \geq 2$,

$$\mathbb{P}(Y \geq k) \leq \frac{\mathbb{E}Y}{k-1} - \frac{1}{k-1}.$$

If we now apply this inequality to Y as above, we find

$$\mathbb{P}(Y \geq 3) \leq \frac{2}{2} - \frac{1}{2} = \frac{1}{2}.$$

This matches the bound given by Markov's inequality when applied to $\mathbb{P}(X \geq 2)$.

Overall, we find that Markov's inequality gives a worse bound when applied to a random variable which has a minimum strictly larger than zero. In this case, we can simply shift the random variable to have a minimum value of zero, and then apply Markov's inequality to get a better bound.